THE COMPLEX LORENTZIAN LEECH LATTICE AND THE BIMONSTER (II)

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ABSTRACT. Let D be the incidence graph of the projective plane over $\mathbb{F}_3$. The Artin group of the graph D maps onto the bimonster and a complex hyperbolic reflection group Γ acting on 13 dimensional complex hyperbolic space Y. The generators of the Artin group are mapped to elements of order 2 (resp. 3) in the bimonster (resp. Γ). Let $Y^\circ \subseteq Y$ be the complement of the union of the fixed points of reflections in Γ. Daniel Allcock has conjectured that the orbifold fundamental group of Y°/Γ surjects onto bimonster.

In this article we study the reflection group Γ. We show that the Artin group of D maps to the orbifold fundamental group of Y°/Γ, thus answering a question in Allcock's article "A monstrous proposal" and taking one step towards the proof of Allcock's conjecture. The finite group $\mathrm{Aut}(D)$ acts on Y. We make a detailed study of the complex hyperbolic line fixed by the subgroup $\mathrm{L}_3(3) \subseteq \mathrm{Aut}(D)$. We define meromorphic automorphic forms on Y, invariant under Γ, with poles along mirrors. We show that the restriction of these forms to the complex hyperbolic line fixed by $\mathrm{L}_3(3)$ gives meromorphic modular forms of level 13.

1. Introduction

This article is a continuation of [4]. Here we continue our study of the reflection group of the complex Lorentzian Leech lattice. Before describing the new results, we briefly recall the context, which makes the study of this particular reflection group interesting.

1.1. **Background:** We begin with little bit of notation. Let $\mathbb{D}$ be a graph with vertex set $\{x_1, \cdots, x_k\}$. Let $\mathcal{A}(\mathbb{D})$ be the group generated by k generators, also denoted by $x_1, \cdots, x_k$, and the relations

$$x_i x_j = x_j x_i \text{ if } \{x_i, x_j\} \text{ is not an edge of } \mathbb{D},$$
$$x_i x_j x_i = x_j x_i x_j \text{ if } \{x_i, x_j\} \text{ is an edge of } \mathbb{D},$$

for all i and j. The group $\mathcal{A}(\mathbb{D})$ is called the *Artin group* of the graph $\mathbb{D}$. Let $\mathrm{Cox}(\mathbb{D}, n)$ be the quotient of $\mathcal{A}(\mathbb{D})$, obtained by imposing the relations $x_i^n = 1$ for all $x_i \in \mathbb{D}$. In this notation, the Coxeter group of the graph $\mathbb{D}$ is $\mathrm{Cox}(\mathbb{D}, 2)$.

The bimonster is the wreath product of the monster simple group with $\mathbb{Z}/2\mathbb{Z}$. Conway and Norton conjectured a simple presentation of the bimonster that later became the Ivanov-Norton theorem.

Theorem ([12], [13], [15]). *Let M_{666} the graph, shaped like an "Y", with 16 vertices (six in each hand including the central vertex). Label the successive vertices in the i–th hand by*

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$a, b_i, c_i, d_i, e_i, f_i$ with a being the central vertex. Then $\mathrm{Cox}(M_{666}, 2)$ *maps onto the bimonster and the kernel is generated by the single relation* $(ab_1c_1ab_2c_2ab_3c_3)^{10} = 1$.

Using the Ivanov-Norton theorem, Conway et.al. obtained a second presentation of the bimonster. Let D be the incidence graph of the projective plane over the finite field $\mathbb{F}_3$. The graph M_{666} is a maximal sub-tree of D, so there is a natural map from $\mathrm{Cox}(M_{666}, 2)$ to $\mathrm{Cox}(D, 2)$.

Theorem ([9], also see [7], [8])**.** *The surjection from* $\mathrm{Cox}(M_{666}, 2)$ *to the bimonster, extend to a surjection from* $\mathrm{Cox}(D, 2)$ *to the bimonster. The kernel is generated by some explicitly described simple relations called "deflating the* 12*–gons".*

We need a bit of notation to introduce the reflection group. Let $\omega = e^{2\pi i/3}$ and $\mathcal{E} = \mathbb{Z}[\omega]$. Define the $\mathcal{E}$–lattice L to be the direct sum of the complex Leech lattice and a hyperbolic cell (See 2.4-2.5 of [4]). Equivalently, we may define L to be the unique Hermitian lattice defined over $\mathcal{E}$ such that $(2+\omega)L' = L$ (where L' denotes the dual lattice of L). Let $Y = \mathbb{P}_+(L^{\mathbb{C}})$ be the set of complex lines of positive norm in the complex vector space $L^{\mathbb{C}} = L \otimes_{\mathcal{E}} \mathbb{C}$. The space Y is isomorphic to 13 dimensional complex hyperbolic space. The projective automorphism group $\mathbb{P}\,\mathrm{Aut}(L)$ acts faithfully on Y. We write $\Gamma = \mathbb{P}\,\mathrm{Aut}(L)$.

Daniel Allcock showed in [1] that the reflection group of L, denoted by $R(L)$, has finite index in $\mathrm{Aut}(L)$. So $R(L)$ is an arithmetic subgroup of $U(1, 13)$. This example provides the largest dimension in which an arithmetic hyperbolic reflection group is known. Allcock also observed that there is a map $\phi : \mathrm{Cox}(M_{666}, 3) \to R(L)$ sending the generators to complex reflections of order three. While trying to better understand the reflection group of L, we made the following observations:

1.2. **Theorem** ([4])**.** *(a) The map* $\phi : \mathrm{Cox}(M_{666}, 3) \to R(L)$ *is onto and it extends to a surjection* $\phi : \mathrm{Cox}(D, 3) \to R(L)$*. In other words, there are* 26 *complex reflections of order* 3 *(called simple reflections) in the reflection group of* L *which braid or commute according to the diagram* D *and these reflections generate* $R(L)$*. (The proof of this fact, given in* [4] *used a computer calculation. Allcock has recently found a computer free proof; see* [3]*.).*
(b) One has $\mathrm{Aut}(L) = R(L)$*. The "deflation relation" holds in* $R(L)$*.*
(c) (see prop. 6.1 in [4]*) Let* $D^{\perp}$ *be the set of* 26 *mirrors, fixed by the* 26 *simple reflections. The group* Γ *possesses a subgroup* $Q \simeq 2.\,\mathrm{L}_3(3)$ *(called the group of diagram automorphisms) which permutes the mirrors* $D^{\perp}$ *in the same way as* $2.\,\mathrm{L}_3(3)$ *permutes the points and lines of* $\mathbb{P}^2(\mathbb{F}_3)$*. The action of* Q *fixes a unique point* $\bar{\rho}$ *in the complex hyperbolic space* Y*. The set* $D^{\perp}$ *consists of precisely the mirrors that are closest to* $\bar{\rho}$*.*

We are interested in understanding the relationship between $R(L)$ and the bimonster. Daniel Allcock has made a conjecture to explain this relationship using complex hyperbolic geometry. (In-fact the conjecture predates [4]).

1.3. **Conjecture** (Allcock, see [2], [4])**.** *Consider the action of* $\Gamma = \mathbb{P}\,\mathrm{Aut}(L)$ *on the complex hyperbolic space* Y*. Let* $\mathcal{M}$ *be the union of the fixed points of the reflections in* Γ*. Let* $Y^{\circ} = Y \setminus \mathcal{M}$*. Then the orbifold fundamental group of* Y°/Γ *maps onto bimonster.*

See [2] for more on this conjecture and its possible ramifications. There are speculative ideas explored in [2] regarding a possible candidate for the conjectured monster manifold and an interpretation of Y/Γ as a nice moduli space.

1.4. **Summary of results:** In section 2, we recall some basic definitions and set up our notations. We describe a construction of Lorentzian lattices from incidence graphs of finite projective planes. This is a direct generalization of the constructions given in [4] and [5] where two lattices, obtained by this construction, are studied.

Section 3 recalls the results from [4] that we need, in some detail. Lemma 3.2 and theorem 5.8 of [4] show that $\mathrm{Aut}(L)$ can be generated by sixteen complex reflections of order 3, making an M_{666} diagram. The only new result in section 3 is a small improvement of this theorem. We prove that fourteen complex reflections of order 3 suffice to generate the reflection group of L. Since L is 14 dimensional, these fourteen reflections, making a M_{655} diagram, form a minimal set of generators for $\mathrm{Aut}(L)$.

In section 4 we study the fundamental group of the orbifold Y°/Γ, that we denote by G. By definition of G (see 4.3), there is an exact sequence,

$$1 \to \pi_1(Y^\circ) \to G \xrightarrow{\pi_\Gamma^G} \Gamma \to 1. \tag{1}$$

By the results from [4] quoted above, we have a surjection from $\mathcal{A}(D)$ to Γ. By a slight abuse of notation, we again denote this map by ϕ. The main result of this section is theorem 4.4, which we have quoted below. This answers a question asked by Allcock in [2].

Theorem. *There exists a homomorphism $\psi : \mathcal{A}(D) \to G$ such that $\pi_\Gamma^G \circ \psi = \phi$.*

This theorem provides one step in an approach to prove Allcock's conjecture stated above. Let $\tilde{Y}^\circ$ be the universal cover of Y. The group G acts as Deck transformations on the ramified covering $(\tilde{Y}^\circ \to Y^\circ/\Gamma)$. Let $G_1 \subseteq G$ be the image of the map ψ. Let N be the normal subgroup of G_1 generated by $\{\psi(r)^2 : r \in D\}$. Then we have the following tower of ramified covering:

$$\begin{tikzcd} & \tilde{Y}^\circ \arrow[dl] \arrow[dr] & \\ Y^\circ \arrow[dr] & & \tilde{Y}^\circ/N \arrow[dl] \\ & Y^\circ/\Gamma & \end{tikzcd}$$

The following observations should justify the effort needed to prove the theorem above.

(1) By theorem 4.4 (quoted above), there is a surjection from $\mathrm{Cox}(D, 2)$ to G_1/N. If the "deflation relations" hold in G_1/N and G_1/N has more than two elements then G_1/N is the bimonster. On the other hand, G_1/N acts naturally as deck transformations on the (possibly ramified) covering $(\tilde{Y}^\circ/N \to Y^\circ/\Gamma)$.

(2) In addition to the above, if the map ψ is onto, that is, if $G_1 = G$, then that proves Allcock's conjecture. If $G_1 = G$, then the deflation relation probably holds in G/N. Sketch of a geometric argument to prove this was explained to me by Daniel Allcock.

We prove theorem 4.4 by constructing explicit homotopies between paths in Y° to obtain relations in the group G. Let γ_1 and γ_2 be two paths in Y° with the same beginning and endpoints. To construct a homotopy from γ_1 to γ_2 in Y°, we construct a 2–cell $C' \subseteq Y$ whose boundary is $\gamma_1 \cup \gamma_2$ and then show that C' avoids the mirrors of reflection. The main tool to show that C' avoids the mirrors, is to use theorem 1.2(c), which provides some information about the configuration of mirrors near the point $\bar{\rho} \in Y^\circ$ fixed by the group $2.\mathrm{L}_3(3)$ of diagram automorphisms. This information about mirror arrangements, together with some

complex hyperbolic geometry, restrict the possible set of mirrors intersecting C to a finite set. The proof is completed by directly checking that the remaining finite set of mirrors do not intersect C.

The unique point $\bar{\rho}$ fixed by the group $Q \simeq 2.\mathrm{L}_3(3)$ of diagram automorphisms, plays an important role in all the major arguments in [4] as well as the present article. In view of this, in section 5 we make a detailed study of the two dimensional lattice F, that is fixed by the group $\mathrm{L}_3(3) \subseteq Q$. The complex hyperbolic line $\mathbb{P}_+(F^{\mathbb{C}})$ is isomorphic to the upper half plane $\mathcal{H}^2$. We construct an explicit isometry $\beta : \mathbb{P}_+(F^{\mathbb{C}}) \to \mathcal{H}^2$ such that "cusps of Leech type" map to 0 and ∞ and $\bar{\rho}$ maps to $\sqrt{-1}$. Let Γ_F be the set of elements of Γ that fix F as a set. The isometry β induces a group homomorphism $c_\beta : \Gamma_F \to \mathrm{PSL}_2(\mathbb{R})$. We show that the image of c_β contains the principal congruence subgroup $\Gamma(13)$. The group $c_\beta(\Gamma_F) \cap \mathrm{PSL}_2(\mathbb{Z})$ has genus zero. In-fact it is a conjugate of $\Gamma_0(13)$. The diagram automorphism σ, that correspond to interchanging points and lines of $\mathbb{P}^2(\mathbb{F}_3)$, acts as the involution $(\tau \mapsto -\tau^{-1})$.

This result provides an explicit way to obtain ordinary modular forms from automorphic forms of type $U(1,13)$ defined on the hermitian symmetric space Y. Let f be a meromorphic automorphic form on $\mathbb{C}H^{13}$ automorphic with respect to the group Γ, with zeroes and poles along the mirrors of reflection. Then the restriction of f to the complex hyperbolic line $\mathbb{P}_+(F^{\mathbb{C}})$ is a meromorphic modular form of level thirteen. Since F is not contained in any mirror, the restriction of f to $\mathbb{P}_+(F^{\mathbb{C}})$ is a well-defined and non-zero meromorphic function. Meromorphic automorphic forms on Y with zeros and poles along the mirrors of reflection can be constructed by the Borcherds method of singular theta lift (see [6] for the general method and [1] for our example). Alternatively, as we shall see in 5.3, such automorphic forms can be directly defined by infinite series that are easy to write down (analogous to the Poincare or Eisenstein series). These automorphic forms may be useful in obtaining an explicit projective uniformization of Y°/Γ.

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1.5. **Index of some commonly used notations:** We use the Atlas notation for groups.

$\mathcal{A}(D)$	the Artin group of the diagram D.
$\mathrm{Aut}(L)$	automorphism group of the lattice L.
β	an isometry from $\mathbb{P}_+(\mathbb{F}^{\mathbb{C}})$ to $\mathcal{H}^2$ or the matrix $\left(\begin{smallmatrix} p & \omega \\ 1 & \bar{p} \end{smallmatrix}\right)$ that represent it.
$\mathbb{C}H^n$	the complex hyperbolic space of dimension n.
D	the incidence graph of $P^2(\mathbb{F}_3)$ or the set of 26 simple roots of L labeled by vertices of this graph.
$\mathcal{E}$	$= \mathbb{Z}[e^{2\pi i/3}]$.
ϵ	a small positive real number.
F	the sub-lattice of L fixed by the group $\mathrm{L}_3(3)$ of diagram automorphisms.
G	the orbifold fundamental group of X°.
Γ	the automorphism group of L modulo scalars: $\Gamma = \mathbb{P}\,\mathrm{Aut}(L)$.
$\mathrm{ht}(r)$	height of a vector r, given by $\mathrm{ht}(r) = \lvert\langle\bar{\rho}, r\rangle\rvert / \lvert\bar{\rho}\rvert^2$.
l	an element of $\mathcal{L}$.
L	the complex Leech lattice plus a hyperbolic cell, defined over $\mathcal{E}$.
L'	the dual lattice of L.

$\mathcal{L}$	the set of lines of $\mathbb{P}^2(\mathbb{F}_3)$ or the simple roots of L that correspond to them.
$\mathrm{L}_3(3)$	$= \mathrm{PSL}_3(\mathbb{F}_3) = \mathrm{PGL}_3(\mathbb{F}_3)$.
$\mathcal{M}$	the union of the mirrors of the reflection group of L.
p	$= 2 + \omega$.
p_1	$= 3 - \omega$.
$\mathcal{P}$	the set of points of $\mathbb{P}^2(\mathbb{F}_3)$ or the simple roots of L that correspond to them.
ϕ_r	ω–reflection in the vector r.
Q	the group of diagram automorphisms acting on Y; one has $Q \simeq 2.\mathrm{L}_3(3)$.
r_i	$r_1, \cdots, r_{26}$ are the simple roots.
ρ_i	$\rho_i = r_i$ if $r_i \in \mathcal{P}$, and $\rho_i = \xi r_i$ if $r_i \in \mathcal{L}$.
$R(L)$	(complex) reflection group of the lattice L.
$[\bar{\rho}]$	the unique point in the complex hyperbolic space Y, fixed by Q. (see equation (5)).
σ	A diagram automorphism that corresponds to interchanging the points and lines of $\mathbb{P}^2(\mathbb{F}_3)$.
θ	$= \omega - \bar{\omega}$.
ω	$e^{2\pi i/3}$.
$[w_{\mathcal{P}}]$	The point of Y where the 13 mirrors $\{x^{\perp} : x \in \mathcal{P}\}$ meet. (see equation (3))
$[w_{\mathcal{L}}]$	The point of Y where the 13 mirrors $\{l^{\perp} : l \in \mathcal{L}\}$ meet. (see equation (3))
x	an element of $\mathcal{P}$.
ξ	$e^{-\pi i/6}$.
X°	the orbifold Y°/Γ.
Y	the set of complex lines of positive norm in $L \otimes_{\mathcal{E}} \mathbb{C}$. (Note: $Y \simeq \mathbb{C}H^{13}$.)
Y°	$= Y \setminus \mathcal{M}$.

2. Some generalities on complex hyperbolic reflection groups

2.1. **The complex hyperbolic space:** A general reference for complex hyperbolic geometry is [11]. Let V be a complex vector space, with an Hermitian form $\langle\ ,\ \rangle$. Given a basis $b = (b_1, \cdots, b_m)$ for V, let $\mathrm{gram}(V)_b = ((\langle b_i, b_j \rangle))$ be the gram matrix of (V, b). Assume that V is Lorentzian, that is, V has signature $(1, m-1)$. Then the open subset $\mathbb{P}_+(V)$ of the projective space $\mathbb{P}(V)$, consisting of the complex lines of positive norm, is called the complex hyperbolic space of V. If $H \subseteq V$, then let $[H]$ be the subset of $\mathbb{P}_+(V)$ or $\mathbb{P}(V)$ determined by H. We shall often abuse notation and write H for $[H]$, if there is no possibility of confusion.

Let $\mathbb{C}^{m,n}$ denote the complex vector space $\mathbb{C}^{m+n}$ with the Hermitian form

$$\langle z, w \rangle = \sum_{i=1}^{m} \bar{z}_i w_i - \sum_{i=m+1}^{m+n} \bar{z}_i w_i.$$

The n dimensional complex hyperbolic space $\mathbb{C}H^n = \mathbb{P}_+(\mathbb{C}^{1,n})$ is homeomorphic to the unit ball $B^n(\mathbb{C}) \subseteq \mathbb{C}^n$ via the isomorphism $b' : [z_0, \cdots, z_n]' \mapsto (z_1/z_0, \cdots, z_n/z_0)$. Let x and y be two vectors in V having positive norm. The metric on $\mathbb{C}H^n$ is given by

$$d([x],[y]) = \cosh^{-1}\Big(\frac{|\langle x, y \rangle|}{|x|.|y|}\Big).$$

2.2**. The complex hyperbolic line and the real hyperbolic plane:** We need to consider two models of the real hyperbolic plane, namely the unit ball $B^1(\mathbb{C})$ and the upper half plane $\mathcal{H}^2$. It will be convenient to identify both as subsets of $\mathbb{P}^1(\mathbb{C})$ by $\tau \mapsto \left(\begin{smallmatrix}\tau\\1\end{smallmatrix}\right)$.

The map $b' : [z_0, z_1] \to z_1/z_0$ defines an isometry between the complex hyperbolic line $\mathbb{C}H^1$ and the (ball model of) real hyperbolic plane. Let $\mathcal{H}^2$ be the upper half plane with the Poincare metric: $d(\tau,\tau') = \cosh^{-1}\bigl(2^{-1}\operatorname{Im}(\tau)^{-\frac{1}{2}}\operatorname{Im}(\tau')^{-\frac{1}{2}}|\tau' - \bar{\tau}|\bigr)$. Let $C : B^1(\mathbb{C}) \to \mathcal{H}^2$ denote the Cayley isomorphism: $C(w) = i\bigl(\frac{1+w}{1-w}\bigr)$. The composition

$$b = C \circ b' : \mathbb{P}_+(\mathbb{C}^{1,1}) \to \mathcal{H}^2$$

is an isometry. Let $S = \left(\begin{smallmatrix}0 & -1\\1 & 0\end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix}1 & 1\\0 & 1\end{smallmatrix}\right)$ be the standard generators of $\mathrm{SL}_2(\mathbb{Z})$.

2.3**. Complex lattices and their reflection groups:** We recall our notations regarding complex reflection groups. In most places, we maintain the notations of [4] and refer the reader to section 2.2 of [4] for more details. Let $\xi = e^{-\pi i/6}$, $\omega = -\xi^2$ and $\mathcal{E} = \mathbb{Z}[\omega]$. An *$\mathcal{E}$–lattice* K is a free $\mathcal{E}$–module of finite rank, together with an $\mathcal{E}$–valued Hermitian form $\langle\ ,\ \rangle : K \times K \to \mathcal{E}$ (always linear in the second variable). Let $K^{\mathbb{C}} = K \otimes_{\mathcal{E}} \mathbb{C}$ be the underlying Hermitian vector space of K. We shall mainly be concerned with a lattice L, for which $L^{\mathbb{C}} \simeq \mathbb{C}^{1,13}$. So the definite lattices that we consider will always be negative definite. Let K' be the *dual lattice* of K, defined by $K' = \{v \in K^{\mathbb{C}} \colon \langle v, x\rangle \in \mathcal{E} \text{ for all } x \in K\}$. Let $\mathrm{D}(K) = K'/K$ be the *discriminant group* of K.

Let r be a primitive vector of K having negative norm, that is, $|r|^2 := \langle r, r\rangle < 0$. Let α be a root of unity in $\mathcal{E}$, $\alpha \neq 1$. A *complex reflection* $\phi_r^\alpha \in \mathrm{Aut}(K)$ is an automorphism of K that fixes the hyperplane $r^\perp$ orthogonal to r and multiplies r by α. The vector r is called the *root* of the reflection and the hyperplane $r^\perp$ (or its image in the projective space $\mathbb{P}(K^{\mathbb{C}})$) is called the *mirror* of the reflection. The *reflection group* of K, denoted by $R(K)$, is the subgroup of $\mathrm{Aut}(K)$ generated by reflections in the roots of K. We write $\phi_r = \phi_r^\omega$ and call it the ω–reflection in r.

2.4**. Lorentzian lattice from incidence graph of finite projective plane:** Let $\mathbb{P}^2(\mathbb{F}_q)$ be the projective plane over the finite field $\mathbb{F}_q$. Let $\mathcal{O}$ be an integral domain containing $\mathbb{Z}$ with an involution $z \mapsto \bar{z}$, such that for all $z \in \mathcal{O}$, the norm $|z|^2 = z\bar{z}$ belongs to $\mathbb{Z}$. Suppose that there exists a prime p in $\mathcal{O}$ such that $|p|^2 = q$. Let $n = q^2 + q + 1$. Given this data we shall construct a Hermitian $\mathcal{O}$–lattice L_p of signature $(1, n)$. The lattice L that we want to study in this article is obtained when $\mathcal{O} = \mathcal{E}$ and $p = 2 + \omega$.

Let $\mathcal{P}$ be the set of points and $\mathcal{L}$ be the set of lines of $\mathbb{P}^2(\mathbb{F}_q)$. The sets $\mathcal{P}$ and $\mathcal{L}$ have n elements each. If a point $x \in \mathcal{P}$ is incident on a line $l \in \mathcal{L}$, then we write $x \in l$. Let D be the (directed) incidence graph of $\mathbb{P}^2(\mathbb{F}_q)$. The vertex set of D is $\mathcal{P} \cup \mathcal{L}$. There is a directed edge in D from a vertex l to a vertex x if $x \in \mathcal{P}$, $l \in \mathcal{L}$ and $x \in l$.

Let L_p° be the singular $\mathcal{O}$–lattice of rank $2n$ with basis vectors indexed by $D = \mathcal{P} \cup \mathcal{L}$. Let $x, x' \in \mathcal{P}$ and $l, l' \in \mathcal{L}$. The inner product on L_p° is defined by

$$\langle x, x'\rangle = \begin{cases} -q & \text{if } \ x = x', \\ 0 & \text{otherwise.}\end{cases} \quad \langle l, l'\rangle = \begin{cases} -q & \text{if } \ l = l', \\ 0 & \text{otherwise.}\end{cases} \quad \langle x, l\rangle = \begin{cases} p & \text{if } \ x \in l, \\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

2.5. **Lemma.** *Given $l \in \mathcal{L}$, let $w_l = \bar{p}l + \sum_{x \in l} x$. Then $\langle x', w_l\rangle = 0$ and $\langle l', w_l\rangle = \bar{p}$ for all $x' \in \mathcal{P}$ and $l' \in \mathcal{L}$.*

Proof. The inner products are easily calculated from (2). If $x' \in \mathcal{P}$ is incident on l, then $\langle x', w_l \rangle = \bar{p}p - q = 0$. If x' is not incident on l, then again, $\langle x', w_l \rangle = 0$. Let $l' \in \mathcal{L}$. If $l' = l$, then $\langle l', w_l \rangle = \bar{p}(-q) + \bar{p}(q+1) = \bar{p}$, because each line has $(q+1)$ points incident on it. If $l' \neq l$, then there is a unique point incident on both l and l'. This implies $\langle l', w_l \rangle = \bar{p}$. ☐

2.6. **Definition.** If l_1 and l_2 are two distinct elements of $\mathcal{L}$, then lemma 2.5 implies that $(w_{l_1} - w_{l_2})$ is orthogonal to each basis vector of L_p°. So $(w_{l_1} - w_{l_2})$ has norm zero. Let U be the sub-lattice spanned by $(w_{l_1} - w_{l_2})$ for all $l_1, l_2 \in \mathcal{L}$. Define $L_p = L_p^{\circ}/U$. The singular lattice U is $(n-1)$ dimensional. So L_p is a $(n+1)$ dimensional $\mathcal{O}$–module. Since U consists of norm zero vectors, the Hermitian form on L_p° descends to a Hermitian form on L_p, which is again denoted by $\langle\ ,\ \rangle$. Let $w_{\mathcal{P}}$ be the image of w_l in L_p (for any $l \in \mathcal{L}$). Lemma 2.5 implies that the vector $w_{\mathcal{P}}$ is orthogonal (in L_p) to each $x \in \mathcal{P}$ and $\langle w_{\mathcal{P}}, l \rangle = p$ for all $l \in \mathcal{L}$. Note that

$$\langle w_{\mathcal{P}}, w_{\mathcal{P}} \rangle = \langle \bar{p}l + \sum_{x \in l} x, w_{\mathcal{P}} \rangle = p \langle l, w_{\mathcal{P}} \rangle = q.$$

Let $\mathfrak{F}$ be the fraction field of $\mathcal{O}$. We see that the gram matrix of $L_p^{\mathfrak{F}}$ with respect to the basis $\mathcal{P} \cup \{w_{\mathcal{P}}\}$ is a diagonal matrix with diagonal entries $(-q, -q, \cdots, -q, q)$. So L_p is a non-degenerate $\mathcal{O}$–module of signature $(1, n)$.

3. The reflection group of the Lorentzian Leech lattice

We have already stated the main results of [4] in the introduction. Since all our work build on them, we repeat these results here in some more detail. Then we prove a couple of small lemmas. As mentioned in the introduction, let L be the direct sum of the complex Leech lattice and a hyperbolic cell.

3.1**. Observation:** *The lattice L contains 26 vectors of norm -3 (called the simple roots) that have inner products prescribed by (2), with $p = 2 + \omega$ (see 3.1 of [4]). So $L \simeq L_p$, where L_p is defined in 2.6.*

We want to study the action of $\Gamma = \mathbb{P}\operatorname{Aut}(L)$ on $Y = \mathbb{P}_+(L^{\mathbb{C}}) \simeq \mathbb{C}H^{13}$. Note that the simple roots of L correspond to the vertices of the graph D. A vertex of D and the corresponding simple root of L are denoted by the same symbol. If $\{r, s\}$ is an edge of D, then the ω–reflections in the simple roots r and s braid. If $\{r, s\}$ is not an edge, then the reflections commute. As mentioned in 1.2(a), the order three complex reflections in the 26 simple roots generate $\operatorname{Aut}(L)$.

The construction of L given in 2.6 implies that the group of *diagram automorphisms* $Q \simeq 2.\,\mathrm{L}_3(3)$ acts on the complex hyperbolic space $Y = \mathbb{P}_+(L^{\mathbb{C}})$. The action of $\mathrm{L}_3(3)$ on L point-wise fixes a two dimensional primitive sub-lattice F of L having a basis $w = (w_{\mathcal{P}}, w_{\mathcal{L}})$ where

$$w_{\mathcal{P}} = \bar{p}l + \sum_{x' \in l} x' \quad \text{and} \quad w_{\mathcal{L}} = px + \sum_{x \in l'} l', \tag{3}$$

for any $x \in \mathcal{P}$ and $l \in \mathcal{L}$. We shall make a detailed study of the lattice F in section 5. From the inner products given in (2), it is easy to check that for all $x \in \mathcal{P}$ and $l \in \mathcal{L}$, we have,

$$\langle w_{\mathcal{P}}, x \rangle = \langle w_{\mathcal{L}}, l \rangle = 0,\ \langle w_{\mathcal{P}}, l \rangle = \langle x, w_{\mathcal{L}} \rangle = p, \quad \text{and} \quad \operatorname{gram}(F)_w = \left(\begin{smallmatrix} 3 & 4p \\ 4\bar{p} & 3 \end{smallmatrix}\right). \tag{4}$$

Let $D^{\perp}$ be the set of mirrors perpendicular to the simple roots. These are called the *simple mirrors*. Equation (4) implies that the thirteen simple mirrors in $\mathbb{C}H^{13}$ corresponding to

the elements of $\mathcal{P}$ (resp. $\mathcal{L}$) meet at $[w_{\mathcal{P}}]$ (resp. $[w_{\mathcal{L}}]$). There exists $\sigma \in \mathrm{Aut}(L)$ (cf. [4], Section 5.3) that corresponds to interchanging points and lines of $\mathbb{P}^2(\mathbb{F}_3)$. The group $Q \subseteq \Gamma$ of diagram automorphisms is generated by σ and $\mathrm{L}_3(3)$. Let

$$\bar{\rho} = \tfrac{1}{26}\Big(\sum_{x \in \mathcal{P}} x + \xi \sum_{l \in \mathcal{L}} l\Big) = \frac{w_{\mathcal{P}} + \xi w_{\mathcal{L}}}{2(4+\sqrt{3})}. \tag{5}$$

Then $[\bar{\rho}]$ is the midpoint of the geodesic joining $[w_{\mathcal{P}}]$ and $[w_{\mathcal{L}}]$. The diagram automorphism σ interchanges $[w_{\mathcal{P}}]$ and $[w_{\mathcal{L}}]$, so $[\bar{\rho}]$ is the only point in $\mathbb{C}H^{13}$ fixed by Q. Since Q fixes $[\bar{\rho}]$ and acts transitively on the simple mirrors, it follows that $[\bar{\rho}]$ is equidistant from the simple mirrors. Let $d_0 = d(r^{\perp}, [\bar{\rho}])$ for $r \in D$. As mentioned in theorem 1.2(c), we know that $d(r^{\perp}, [\bar{\rho}]) \geq d_0$ for all root r of L and $d(r^{\perp}, [\bar{\rho}]) = d_0$ if and only if $r^{\perp}$ is a simple mirror. In other words, the simple mirrors are closest to $[\bar{\rho}]$ and all other mirrors are further away.

3.2. **Lemma.** *Consider $\Gamma = \mathbb{P}\,\mathrm{Aut}(L)$ acting on $Y \simeq \mathbb{C}H^{13}$. Then the stabilizer (in Γ) of the set $D^{\perp}$ and the stabilizer of $[\bar{\rho}]$ are both equal to $Q \simeq 2.\,\mathrm{L}_3(3)$.*

Proof. If $g \in \mathbb{P}\,\mathrm{Aut}(L)$ fixes $D^{\perp}$ as a set, then it must fix $[\bar{\rho}]$, it being the only point, equidistant from each simple mirror. Conversely, if $g \in G$ fixes $[\bar{\rho}]$ then it must permute the mirrors in $D^{\perp}$ in some way and thus determine an element of $2.\,\mathrm{L}_3(3)$. So the stabilizer of the set $D^{\perp}$ and the point $[\bar{\rho}]$ are equal.

Suppose g acts trivially on $D^{\perp}$. Let $\tilde{g}$ be a lift of g in $\mathrm{Aut}(L)$. For each $r \in D$, it follows that $\tilde{g}r = \mu_r r$ for some root of unity μ_r. If r and s are two simple roots with $\langle r, s\rangle = p$, then $p = \langle \tilde{g}r, \tilde{g}s\rangle = \bar{\mu}_r \mu_s \langle r, s\rangle = \bar{\mu}_r \mu_s p$. So $\mu_r = \mu_s$ whenever there is an edge between r and s in the graph D, which implies $\mu_r = \mu$ is a constant. So g is equal to the identity. ☐

In [4], we showed that $\mathrm{Aut}(L)$ is generated by the 16 simple reflections in the roots $\{a, b_j, c_j, d_j, e_j, f_j \colon j = 1, 2, 3\}$ that form the M_{666} diagram. We shall end this section by noting a small improvement of this fact.

3.3. **Lemma.** *The automorphism group of L is generated by the fourteen ω–reflections in the simple roots a, f_1 and b_i, c_i, d_i, e_i for $i = 1, 2, 3$. Since L is 14 dimensional, the 14 simple reflections of a M_{655} diagram form a minimal set of generators for* $\mathrm{Aut}(L)$.

The proof of the lemma uses the deflation relations, which we now recall. By a sub-graph of D we mean the graph formed by taking a subset of vertices of D and all the edges between these vertices in D. A 12–gon in D is a sub-graph of D that has the shape of a circuit of length 12 (in other words, an affine A_{11} diagram), Let y be a 12–gon in D and let $\{y_1, \cdots, y_{12}\}$ be the successive vertices of y. Consider the relation

$$(y_1 y_2 \cdots y_{10}) y_{11} (y_1 y_2 \cdots y_{10})^{-1} = y_{12} \tag{6}$$

Following [9], we call this relation "deflate(y)". The affine Coxeter group of type A_{11} generated by $y_1, \cdots, y_{12}$ reduces to the symmetric group S_{12} in presence of the relation deflate(y). The bimonster is the quotient of $\mathrm{Cox}(D, 2)$ obtained by adding the relations deflate(y) for all free 12–gon y in D (see [9]).

Proof of lemma 3.3. Take the 12–gon $(y_1, \cdots, y_{12}) = (f_2, e_2, d_2, c_2, b_2, a, b_1, c_1, d_1, e_1, f_1, a_3)$ in D. (Here, only for this proof, we are using the names for the simple roots given in [9].) One can check that

$$\phi_{f_2}\phi_{e_2}\phi_{d_2}\phi_{c_2}\phi_{b_2}\phi_a\phi_{b_1}\phi_{c_1}\phi_{d_1}\phi_{e_1}(f_1) = \omega^2 a_3.$$

So equation (6) holds for this 12–gon. It is an amusing exercise to show that the group $2.\,\mathrm{L}_3(3)$ acts transitively on the set of marked 12–gons in D. So the relation (6) holds for any 12–gon in D. The deflation relation implies that

$$y_{12} = (y_1 \cdots y_{10})y_{11}(y_1 \cdots y_{10})^{-1} = (y_{11} \cdots y_2)y_1(y_{11} \cdots y_2)^{-1}.$$

Moving the y_1's to the other side of the equation, and using the commuting relations, we get

$$\begin{aligned}(y_2 \cdots y_{10})y_{11}(y_2 \cdots y_{10})^{-1} &= y_1^{-1}(y_{11} \cdots y_2)y_1(y_{11} \cdots y_2)^{-1}y_1 \\ &= (y_{11} \cdots y_3)y_1^{-1}y_2y_1y_2^{-1}y_1(y_{11} \cdots y_3)^{-1}.\end{aligned}$$

Using the braiding relations $y_1y_2y_1 = y_2y_1y_2$ and $y_i^{-1} = y_i^2$ we get

$$y_1^{-1}y_2y_1y_2^{-1}y_1 = y_1^{-1}y_2y_1y_2y_2y_1 = y_1^{-1}y_1y_2y_1y_2y_1 = y_2y_1y_2y_1 = y_2^{-1}y_1y_2.$$

It follows that

$$(y_{11} \cdots y_3)y_2^{-1}y_1y_2(y_{11} \cdots y_3)^{-1} = (y_2 \cdots y_{10})y_{11}(y_2 \cdots y_{10})^{-1}.$$

So y_1 can be expressed in terms of $y_2, \cdots, y_{11}$. So the reflections ϕ_{f_2} and ϕ_{f_3} can be expressed in terms of of the fourteen reflections stated in the lemma. □

4. The fundamental group of the mirror complement-quotient

In this section we shall show that the Artin group $\mathcal{A}(D)$ maps to the orbifold fundamental group of Y°/Γ. So there is an action of $\mathcal{A}(D)$ on the universal cover of Y° and an action of $\mathrm{Cox}(D, 2)$ on a (possibly ramified) cover of Y°/Γ.

4.1**. Some basics on complex hyperbolic space for estimating distances:** For this sub-section, we let V be a general $(n+1)$ dimensional complex vector space with a Hermitian form of signature $(1, n)$. At the end of the sub-section we shall go back to our example. Let V^+ be the set of vectors of strictly positive norm in V and $\mathbb{P}_+(V) \simeq \mathbb{C}H^n$ be the complex hyperbolic space of V. Whenever we talk of distance in this section, it is with reference to the metric on $\mathbb{C}H^n$. Given x, y in $\mathbb{C}H^n$, let $\mathrm{Geod}(x, y)$ be the real geodesic segment joining x and y.

In the following, let $x_1, \cdots, x_n, x, y, z, w$ be elements of V^+. Assume further that $x_1, \cdots, x_n$ lie in a totally real subspace, that is, $\langle x_i, x_j\rangle \in \mathbb{R}_+$ for $1 \le i, j \le n$. Then, for all $i \neq j$ the Euclidean straight line segment in V^+ joining x_i and x_j (denoted by $\mathrm{Conv}(x_i, x_j)$) determines the real geodesic segment in $\mathbb{C}H^n$ joining x_i and x_j, that is, $[\mathrm{Conv}(x_i, x_j)] = \mathrm{Geod}([x_i], [x_j])$. Let $\mathrm{Conv}(x_1, \cdots, x_n)$ be the convex hull of $x_1, \cdots, x_n$ in V^+:

$$\mathrm{Conv}(x_1, \cdots, x_n) = \{t_1x_1 + \cdots + t_nx_n \colon 0 \le t_j \le 1, \sum t_j = 1\}.$$

The set $\mathrm{Conv}(x_1, \cdots, x_n)$ determines a totally real, totally geodesic subset of $\mathbb{C}H^n$. Given two non-empty subsets Z and W of $\mathbb{C}H^n$ with W compact, let

$$\mathrm{md}_Z(W) = \max\{d(Z, w) \colon w \in W\}.$$

The following observations will be useful for our computation. (Recall our assumption: $\langle x_i, x_j\rangle \in \mathbb{R}_+$).

(1) Let $\{W_j : j \in J\}$ be a (possibly infinite) collection of compact sets such that $\cup_{j\in J}W_j$ is compact. Then

$$\mathrm{md}_H(\cup_{j\in J}W_j) = \sup\{\mathrm{md}_H(W_j) \colon j \in J\} \tag{7}$$

(2) By general properties of negatively curved spaces, one knows that $\max\{d(z,w)\colon w \in \mathrm{Conv}(x_1,x_2)\}$ is attained when w is either x_1 or x_2, in other words

$$\mathrm{md}_z(\mathrm{Conv}(x_1,x_2)) = \max\{d(z,x_1), d(z,x_2)\}.$$

(3) Let $W = \mathrm{Conv}(x_2,\cdots,x_n)$. Using $\mathrm{Conv}(x_1,x_2,\cdots,x_n) = \cup_{w\in W}\mathrm{Conv}(x_1,w)$, it follows that

$$\mathrm{md}_z(\mathrm{Conv}(x_1,\cdots,x_n)) = \sup\{\mathrm{md}_z(\mathrm{Conv}(x_1,w))\colon w \in W\} = \max\{d(z,x_1), \mathrm{md}_z(W)\}.$$

By induction on n, it follows that $\max\{d(z,w)\colon w \in \mathrm{Conv}(x_1,x_2,\cdots,x_n)\}$ is attained when $w = x_i$ for some i.

(4) Let H be a subset of $\mathbb{C}H^n$ and $\Delta = \mathrm{Conv}(x_1,x_2,x_3)$. Let

$$\delta_1 = d(x_1,H) + \max\{d(x_1,x_2), d(x_1,x_3)\}.$$

Define δ_2, δ_3 similarly by cyclic permutation of x_1,x_2,x_3. Choose $w_0 \in \Delta$ be such that $\mathrm{md}_H(\Delta) = d(w_0,H)$. Then, from the previous remarks, we have,

$$\mathrm{md}_H(\Delta) = d(w_0,H) \le d(x_1,H) + d(x_1,w_0) \le d(x_1,H) + \mathrm{md}_{x_1}(\Delta) \le \delta_1.$$

It follows that

$$\mathrm{md}_H(\Delta) \le \min\{\delta_1,\delta_2,\delta_3\}. \tag{8}$$

(5) Let $x \in V^+$ and H be a complex linear subspace in V which meets V^+. Then H determines a totally geodesic subspace of $\mathbb{C}H^n$. Let $\mathrm{pr}_H(x)$ be the "projection" of $[x]$ on H, that is, the point on $[H]$ that is closest to $[x]$.

Let $r_1,\cdots,r_k \in V$ be linearly independent vectors of negative norm and let $H = r_1^\perp \cap \cdots \cap r_k^\perp$. Assume that $H \cap V^+ \neq \emptyset$, in other words, the span of $\{r_1,\cdots,r_k\}$ is negative definite. Then $[H]$ determines a totally geodesic subspace of $\mathbb{C}H^n$ and one has

$$\mathrm{pr}_H(x) = (x + \mathbb{C}r_1 + \cdots + \mathbb{C}r_k) \cap H.$$

4.2**. The setup:** Let L be the lattice already encountered in the introduction and section 3 (the direct sum of the Complex Leech lattice and a hyperbolic cell). Let $Y = \mathbb{P}_+(L^{\mathbb{C}}) \simeq \mathbb{C}H^{13}$ and $\mathcal{M}$ be the union of the mirrors of the reflection group of L. Let $Y^\circ = Y \setminus \mathcal{M}$ be the complement of the mirrors and let $X^\circ = Y^\circ/\Gamma$. A continuous function $\gamma : [0,1] \to Y^\circ$ will be called a path in Y°. Let $\Pi_{Y^\circ}(a,b)$ be the set of homotopy class of paths in Y° beginning at a and ending at b. Given two paths γ and γ' in Y° with $\gamma(1) = \gamma'(0)$, let $\gamma * \gamma'$ be the path obtained by first following γ and then following γ', at double the speed. Let γ_1 and γ_2 be two paths in Y° with same beginning and endpoints. If γ_1 and γ_2 are homotopic in Y°, then we write $\gamma_1 \sim \gamma_2$. The homotopy class of a path γ is denoted by $[\gamma]$.

4.3. **Definition.** Let $G = \{(\gamma,t)\colon t \in \Gamma, \gamma \in \Pi_{Y^\circ}(\bar\rho, t\bar\rho)\}$. Then G becomes a group, with multiplication defined by

$$(\gamma,t).(\gamma',t') = (\gamma * t\gamma', tt').$$

Define $\pi_\Gamma^G : G \to \Gamma$ by $\pi_\Gamma^G(\gamma,t) = t$. The kernel of the epimorphism π_Γ^G is $\pi_1(Y^\circ,\bar\rho)$. So we have an exact sequence

$$1 \to \pi_1(Y^\circ) \to G \xrightarrow{\pi_\Gamma^G} \Gamma \to 1.$$

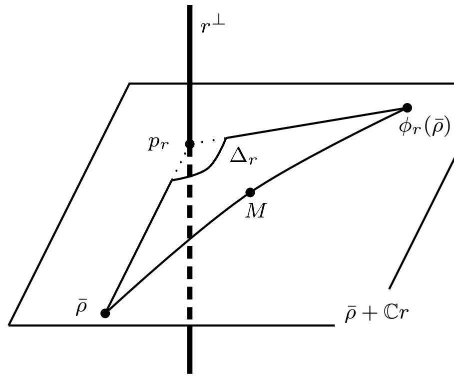

FIGURE 1. a totally geodesic triangle Δ_r such that $\Delta_r \setminus \{p_r\}$ does not intersect any mirror.

Let $\tilde{Y}^\circ = \cup_{y \in Y^\circ} \Pi_{Y^\circ}(\bar\rho, y)$ be the universal cover of Y°. An element of $\tilde{Y}^\circ$ lying above $y \in Y^\circ$ is represented by a path $\lambda : [0,1] \to Y^\circ$ such that $\lambda(1) = y$. The group G acts on $\tilde{Y}^\circ$ by

$$([\gamma], t)[\lambda] = [\gamma * t\lambda].$$

Let $\pi^{\tilde{Y}^\circ}_{X^\circ} : \tilde{Y}^\circ \to X^\circ$ be the projection given by $[\lambda] \mapsto \Gamma\lambda(1)$. Then $\pi^{\tilde{Y}^\circ}_{X^\circ}(g[\lambda]) = \pi^{\tilde{Y}^\circ}_{X^\circ}([\lambda])$ for all $g \in G$, that is, G acts as deck transformations on the ramified covering $\tilde{Y}^\circ \to X^\circ$. For the purpose of this article, we define G to be *orbifold fundamental group* of X°.

By the results mentioned in section 3, there is an surjection $\phi : \mathcal{A}(D) \to \Gamma$, taking the generators of $\mathcal{A}(D)$ to order three simple reflections in Γ. We want to prove the following theorem.

4.4. **Theorem.** *There exists a homomorphism $\psi : \mathcal{A}(D) \to G$ such that $\pi^G_\Gamma \circ \psi = \phi$.*

To give a flavor of the argument, we first prove an easy model lemma.

4.5. **Lemma.** *Let r be a simple root of L. Let p_r be the projection of $\bar\rho$ on $r^\perp$. Let Δ_r be the closed totally geodesic triangle in Y with vertices at $\bar\rho$, $p_r = \mathrm{pr}_{r^\perp}(\bar\rho)$ and $\phi_r(\bar\rho)$ (see figure 1). Then $\Delta_r \setminus \{p_r\}$ does not intersect any mirror.*

(b) Let γ and γ' be any two paths lying in $\Delta_r \setminus \{p_r\}$, starting at $\bar\rho$ and ending at $\phi_x(\bar\rho)$. Then $\gamma \sim \gamma'$ in Y°.

Proof. The isosceles triangle Δ_r is totally geodesic, since it lies in the complex geodesic $\mathbb{P}_+(\mathbb{C}r + \mathbb{C}\bar\rho)$, containing $\bar\rho$ and r. Let M be the midpoint of $\bar\rho$ and $\phi_r(\bar\rho)$. Let Δ^1_r be the triangle with vertices $\bar\rho, p_r, M$ and Δ^2_r be the triangle with vertices $\phi_r(\bar\rho), p_r, M$. So $\Delta_r = \Delta^1_r \cup \Delta^2_r$. One checks that $d(\bar\rho, p_r) > d(\bar\rho, M)$. So the point of Δ^1_r that is furthest from $\bar\rho$ is p_r. But $d(\bar\rho, p_r)$ is the minimum distance between $\bar\rho$ and any mirror (by 1.2(c)). Hence no mirror can intersect $\Delta^1_r \setminus \{p_r\}$. The same statement holds for Δ^2_r by symmetry. This proves part (a). Part (b) follows from part (a). ∎

4.6. **Definition.** Maintain the notations of lemma 4.5. Let r be a simple root. Let $[\gamma]_r$ be the unique homotopy class of paths lying in $\Delta \setminus \{p_r\}$ starting at $\bar\rho$ and ending at $\phi_x(\bar\rho)$.

Define $g_r \in G$, called the *braid reflection* in r, by $g_r = ([\gamma]_r, \phi_r^\omega)$. If $r_1, r_2, \cdots, r_{26}$ are the simple roots, then we write $\phi_i = \phi_{r_i}^\omega$, $[\gamma]_i = [\gamma]_{r_i}$ and $g_i = g_{r_i}$.

Now, theorem 4.4 follows from the following result.

4.7. **Theorem.** *Let r_1 and r_2 be two simple roots of L. If the ω–reflections ϕ_1 and ϕ_2 braid (resp. commute) in Γ, then the braid reflections g_1 and g_2 braid (resp. commute) in G. So $\psi(r_i) = g_i$ defines a homomorphism $\psi : \mathcal{A}(D) \to G$ such that $\pi_\Gamma^G \circ \psi = \phi$.*

The rest of this section is devoted to proving theorem 4.7.

Sketch of proof: We need to set up some notations to make our way smooth.

Let $r_1, \cdots, r_{26}$ be the simple roots. Define the vectors $D_\rho = \{\rho_1, \cdots, \rho_{26}\}$ by

$$\rho_j = \begin{cases} r_j & \text{if } r_j \in \mathcal{P}, \\ \xi r_j & \text{if } r_j \in \mathcal{L}. \end{cases}$$

Sometimes it is more convenient to use the vectors ρ_j instead of r_j. Recall that for all i and j, $\langle \rho_i, \rho_j \rangle$ is a non-negative real number, $\bar{\rho} = \sum_{i=1}^{26} \rho_i/26$ and $\langle \rho_i, \bar{\rho} \rangle = |\bar{\rho}|^2$.

Let r_1 and r_2 be any two distinct simple roots and ρ_1 and ρ_2 be the corresponding elements of D_ρ. Let

$$p_i = \bar{\rho} + \frac{|\bar{\rho}|^2}{3}\rho_i \quad \text{and} \quad q = \bar{\rho} + \frac{|\bar{\rho}|^2}{\alpha}(\rho_1 + \rho_2), \quad \text{where } \alpha = \begin{cases} 3 & \text{if } \langle \rho_1, \rho_2 \rangle = 0 \\ 3 - \sqrt{3} & \text{if } \langle \rho_1, \rho_2 \rangle = \sqrt{3} \end{cases} \tag{9}$$

Then $[p_1]$, $[p_2]$ and $[q]$ are the projections of $\bar{\rho}$ on $\rho_1^\perp$, $\rho_2^\perp$ and $\rho_1^\perp \cap \rho_2^\perp$ respectively.

Suppose the reflections in r_1 and r_2 braid. Let $\phi_1 = \phi_{r_1}^\omega$ and $\phi_2 = \phi_{r_2}^\omega$. We define vectors $z_0, \cdots, z_6$ as follows (see figure 2):

$$z_0 = \bar{\rho}, \;\; z_1 = p_1, \;\; z_2 = \phi_1(\bar{\rho}), \;\; z_3 = \phi_1(p_2), \;\; z_4 = \phi_1\phi_2(\bar{\rho}), \;\; z_5 = \phi_1\phi_2(p_1), \;\; z_6 = \phi_2\phi_1\phi_2(\bar{\rho}).$$

Note that $\langle z_{j-1}, z_j \rangle \in \mathbb{R}_+$, for $j = 1, \cdots, 6$. So the geodesic segment joining z_{j-1} and z_j in Y follows the curve $\mathrm{Conv}(z_{j-1}, z_j)$. We shall use complex hyperbolic geometry to prove the following lemma.

4.8. **Lemma.** *Let $\sigma_1 = \cup_{j=1}^6 \mathrm{Conv}(z_{j-1}, z_j)$. Define σ_2 similarly by interchanging the roles of r_1 and r_2. Then the 2–cell $C = \mathrm{Conv}(q, \sigma_1 \cup \sigma_2) \subseteq V_+$ only intersects the four mirrors $r_1^\perp$, $r_2^\perp$, $\phi_{r_1}(r_2)^\perp$ and $\phi_{r_2}(r_1)^\perp$.*

Note that σ_1 and σ_2 are curves in Y (not in Y°). The curve σ_1 contains the points z_1, z_3 and z_5 which are on the mirrors. So we need to modify σ_1 to avoid this points. For this, fix a small positive real number ϵ. For $j = 1, 2, 3$, let

$$z_{2j-1}^- = z_{2j-1} + \epsilon z_{2j-2} \quad \text{and} \quad z_{2j-1}^+ = z_{2j-1} + \epsilon z_{2j} \tag{10}$$

and let $\tilde{\gamma}_j$ be a path in V_+ that goes from z_{2j-2} to z_{2j} along the curve

$$\mathrm{Conv}(z_{2j-2}, z_{2j-1}^-) \cup \mathrm{Conv}(z_{2j-1}^-, z_{2j-1}^+) \cup \mathrm{Conv}(z_{2j-1}^+, z_{2j}). \tag{11}$$

Define $\sigma_1' = \tilde{\gamma}_1 * \tilde{\gamma}_2 * \tilde{\gamma}_3$ (see figure 2). Similarly define σ_2', by interchanging the role of r_1 and r_2. Lemma 4.5 implies that $[\tilde{\gamma}_1] = [\gamma]_1$. Similarly, $[\tilde{\gamma}_2] = \phi_1[\gamma]_2$ and $[\tilde{\gamma}_3] = \phi_1\phi_2[\gamma]_1$. So $g_1g_2g_1 = ([\sigma_1'], t)$. Similarly $g_2g_1g_2 = ([\sigma_2'], t)$. We need to prove that $\sigma_1' \sim \sigma_2'$. This follows from the lemma given below.

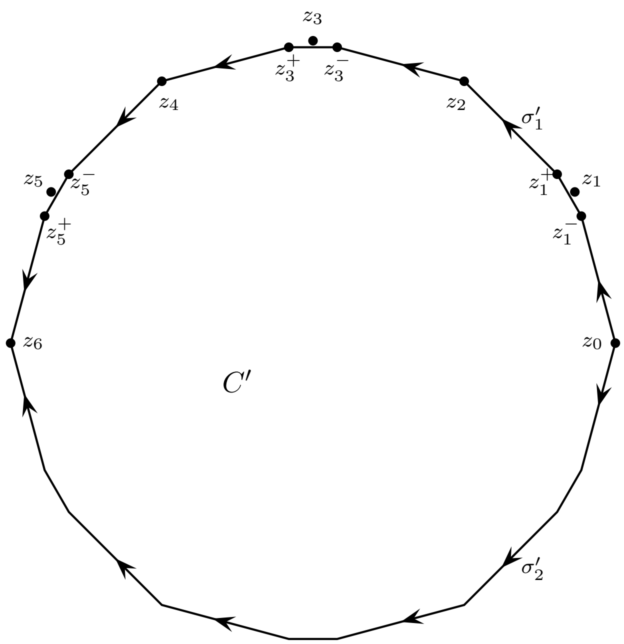

FIGURE 2. Schematic picture of the 2–cell C' and the boundary curves σ_1' and σ_2'. The points z_k and $z_{2j-1}^{\pm}$ on σ_1' are marked.

4.9. **Lemma.** *Let $q' = q + \epsilon\bar{\rho}$. Then the 2–cell $C' = \mathrm{Conv}(q', \sigma_1' \cup \sigma_2') \subseteq V_+$ does not intersect any mirrors.*

Given lemma 4.8, the proof of 4.9 is easy. Since C' is a small perturbation of C, it suffices to check that the four mirrors intersecting C, do not intersect C'. This can be checked by hand. This completes the argument in the case when ϕ_1 and ϕ_2 braids. When ϕ_1 and ϕ_2 commutes, the argument is similar and the calculations are simpler. We briefly indicate the modifications that are needed.

Assume that ϕ_1 and ϕ_2 commute. Let

$$z_0 = \bar{\rho}, \quad z_1 = p_1, \quad z_2 = \phi_1(\bar{\rho}), \quad z_3 = \phi_1(p_2), \quad z_4 = \phi_1\phi_2(\bar{\rho}).$$

Instead of lemma 4.8 we have the following lemma:

4.10. **Lemma.** *Let $\sigma_1 = \cup_{j=1}^4 \mathrm{Conv}(z_{j-1}, z_j)$. Define σ_2 similarly by interchanging the roles of r_1 and r_2. Then the 2–cell $C = \mathrm{Conv}(q, \sigma_1 \cup \sigma_2)$ only intersects the mirrors $r_1^\perp$ and $r_2^\perp$.*

Define $z_{2j-1}^{\pm}$ and $\tilde{\gamma}_j$ for $j = 1, 2$, by the formulas given in (10) and (11). Let $\sigma_1' = \tilde{\gamma}_1 * \tilde{\gamma}_2$. Similarly define σ_2'. With this setting, one has to re-prove lemma 4.9, which amounts to checking that $r_1^\perp$ and $r_2^\perp$ do not intersect C'. $\square$

It remains to prove the lemmas stated in the sketch above.

proof of 4.8 and 4.10. . Let p_1, p_2 and q be as given in (9). Note that the inner products between $\bar{\rho}, p_1, p_2$ and q are all real and they lie in a totally geodesic subspace P of $\mathbb{C}H^{13}$ isomorphic to the real hyperbolic plane. The 2–cell $C \subseteq V^+$ given in 4.8 (resp. 4.10) is a union of 12 (resp. 8) Euclidean triangles and $[C]$ is a union of 12 (resp. 8) totally

geodesic triangles in Y (because q has real inner product with the vertices of σ_1 and σ_2). The boundary of C is $\sigma_1 \cup \sigma_2$.

Consider the quadrilateral $T = \mathrm{Conv}(\bar{\rho}, p_1, q) \cup \mathrm{Conv}(\bar{\rho}, p_2, q)$. If ϕ_1 and ϕ_2 braid, then, with C as given in 4.8, we have,

$$C = T \cup \phi_1(T) \cup \phi_2(T) \cup \phi_1\phi_2(T) \cup \phi_2\phi_1(T) \cup \phi_1\phi_2\phi_1(T).$$

If ϕ_1 and ϕ_2 commute, then, with C as given in 4.10, we have,

$$C = T \cup \phi_1(T) \cup \phi_2(T) \cup \phi_1\phi_2(T).$$

There is a diagram automorphism that interchanges ρ_1 and ρ_2, so the intersection of the triangle $\mathrm{Conv}(\bar{\rho}, p_2, q)$ with the mirrors is exactly similar to that of the triangle $\mathrm{Conv}(\bar{\rho}, p_1, q)$. Lemma 4.8 and 4.10 now follows from lemma 4.11, given below. □

4.11. **Lemma.** *The triangle* $\mathrm{Conv}(\bar{\rho}, p_1, q)$ *meets the mirror* $\rho_1^\perp$ *along the edge* $\mathrm{Conv}(p_1, q)$, *meets* $\rho_2^\perp$ *at* q *and, in the case when* $\langle \rho_1, \rho_2 \rangle = \sqrt{3}$ *it also meets* $\phi_{\rho_1}^{\pm}(\rho_2)^\perp$ *at* q. *Except for these cases no other mirrors intersects* $\mathrm{Conv}(\bar{\rho}, p_1, q) \setminus \{p_1\}$.

proof of lemma 4.11. The proof given below uses some computer verification. These calculations were performed using the GP/PARI calculator. The codes are contained in the file pi1.gp available at www.math.uchicago.edu/˜tathagat/codes/index.html.

The diagonal action of $2.\mathrm{L}_3(3)$ on distinct pairs of simple roots has three orbits:

(1) $\rho_1 \in \mathcal{P}$, $\bar{\xi}\rho_2 \in \mathcal{L}$ and $\langle \rho_1, \rho_2 \rangle = \sqrt{3}$; for calculation we take $\rho_1 = a, \rho_2 = \xi f$.
(2) $\rho_1 \in \mathcal{P}$, $\bar{\xi}\rho_2 \in \mathcal{L}$ and $\langle \rho_1, \rho_2 \rangle = 0$; for calculation we take $\rho_1 = a, \rho_2 = \xi d_1$.
(3) $\rho_1 \in \mathcal{P}$ and $\rho_2 \in \mathcal{P}$; for calculation we take $\rho_1 = a, \rho_2 = c_1$.

Accordingly, we have to consider three cases in the calculations below. If not stated otherwise, the statements below are made for all three cases. It will be convenient to define the *height* of a root r, denoted by $\mathrm{ht}(r)$ as follows:

$$\tilde{\mathrm{ht}}(r) = \langle \bar{\rho}, r \rangle / |\bar{\rho}|^2 \quad \text{and} \quad \mathrm{ht}(r) = |\tilde{\mathrm{ht}}(r)|.$$

Let $d_0 = d(\bar{\rho}, p_1)$. This is the minimum distance from $\bar{\rho}$ to any mirror. Suppose $r^\perp$ is a mirror that intersects the triangle $\Delta_1 = \mathrm{Conv}(\bar{\rho}, p_1, q)$. The longest edge of the triangle Δ_1 is $\mathrm{Conv}(\bar{\rho}, q)$. So

$$\sinh^{-1}\Big(\frac{|\langle r, \bar{\rho} \rangle|}{|r||\bar{\rho}|}\Big) = d(r^\perp, \bar{\rho}) \leq \mathrm{md}_{\bar{\rho}}(\Delta_1) = d(\bar{\rho}, q).$$

This gives a bound for the possible height of the root r:

$$\mathrm{ht}(r) = \frac{|\langle r, \bar{\rho} \rangle|}{|\bar{\rho}|^2} \leq \frac{|r|}{|\bar{\rho}|} \sinh\Big(\cosh^{-1}\Big(\frac{|\langle \bar{\rho}, q \rangle|}{|\bar{\rho}||q|}\Big)\Big) \leq 2.18. \tag{12}$$

Let s be the point on $\mathrm{Conv}(\bar{\rho}, q)$ such that $d(\bar{\rho}, s) = d_0$. The triangle $\mathrm{Conv}(\bar{\rho}, p_1, s)$ cannot meet any mirror except at p_1 and possibly at s. So the mirror $r^\perp$ must intersect the triangle $\Delta = \mathrm{Conv}(p_1, q, s)$. The situation is depicted in the figure 3. One can take $s = \bar{\rho} + c(\rho_1 + \rho_2)$ where $c \in \mathbb{R}_+$ is a constant to be determined. The equality $d(s, \bar{\rho}) = d_0$ implies

$$1 + \frac{|\bar{\rho}|^2}{3} = \frac{|\langle \bar{\rho}, p_1 \rangle|^2}{|\bar{\rho}|^2 |p_1|^2} = \frac{|\langle s, \bar{\rho} \rangle|^2}{|s|^2 |\bar{\rho}|^2} = \frac{|\bar{\rho}|^4 (1 + 2c)^2}{(|\bar{\rho}|^2 + 4c|\bar{\rho}|^2 - 2\alpha c^2)|\bar{\rho}|^2}$$

(where α is given in (9)). Rearranging, one has the quadratic equation

$$(4 + \tfrac{2\alpha}{|\bar{\rho}|^2}(1 + \tfrac{|\bar{\rho}|^2}{3}))c^2 - \tfrac{4}{3}|\bar{\rho}|^2 c - \tfrac{|\bar{\rho}|^2}{3} = 0$$

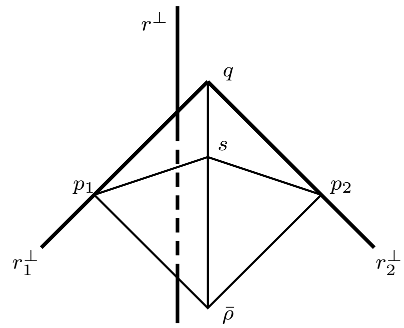

FIGURE 3.

with one positive and one negative root. The positive root gives the required c.

Divide the triangle Δ into two triangles Δ' and Δ'' by joining s with the midpoint of $\mathrm{Conv}(p_1, q)$. For $i = 1, \cdots, 26$, one has,

$$d(\rho_i^{\perp}, r^{\perp}) \leq \mathrm{md}_{\rho_i^{\perp}}(\Delta) = \max\{\mathrm{md}_{\rho_i^{\perp}}(\Delta'), \mathrm{md}_{\rho_i^{\perp}}(\Delta'')\}$$

where the equality is an instance of equation (7) in 4.1. We estimate $\mathrm{md}_{\rho_i^{\perp}}(\Delta')$ and $\mathrm{md}_{\rho_i^{\perp}}(\Delta'')$ using the inequality (8) of 4.1. This gives a bound on $|\langle \rho_i, r\rangle|^2 = 9\cosh^2(d(\rho_i^{\perp}, r^{\perp}))$. By explicit computation, this bound is strictly less than 12 except in three cases when $\langle \rho_1, \rho_2\rangle = \sqrt{3}$ and $\bar{\xi}\rho_i \in \mathcal{L}$. Since the inner product between two vectors of L always lies in $p\mathcal{E}$ one has

$$|\langle x_i, r\rangle|^2 \in \{0, 3, 9\} \quad \text{for } \ i = 1, \cdots, 13. \tag{13}$$

Similarly, from the inequality

$$d(w_{\mathcal{P}}, r^{\perp}) \leq \mathrm{md}_{w_{\mathcal{P}}}(\Delta) \leq \max\{d(w_{\mathcal{P}}, p_1), d(w_{\mathcal{P}}, q), d(w_{\mathcal{P}}, s)\},$$

one gets a bound on $|\langle w_p, r\rangle|^2 = 9\sinh^2(d(r^{\perp}, w_{\mathcal{P}}))$. By explicit computation the bound is strictly less than 10. So one has

$$|\langle w_{\mathcal{P}}, r\rangle|^2 \in \{0, 3, 9\}. \tag{14}$$

The conditions (13) and (14) restrict the possibilities for r to a finite set as in the proof of proposition 6.1 in [4]. Write r in terms of the orthogonal basis $(w_{\mathcal{P}}, x_1, \cdots, x_{13})$ as

$$r = \langle w_{\mathcal{P}}, r\rangle w_{\mathcal{P}}/3 - \sum \langle x_i, r\rangle x_i/3.$$

Taking norm and re-arranging one gets

$$\sum |\langle x_i, r\rangle|^2 = 9 + |\langle w_{\mathcal{P}}, r\rangle|^2$$

which leaves only finitely many possibilities for the vector $(\langle w_{\mathcal{P}}, r\rangle, \langle x_1, r\rangle, \cdots, \langle x_{13}, r\rangle)$. The possible inner products of r with x_i and w_p are shown in table 1. By inspection of the last three entries of fifth column, we find that the minimum height of a root having type 5, 6 or 7, is greater than the bound on $\mathrm{ht}(r)$ obtained in (12). So mirrors of type 5, 6 or 7 cannot intersect Δ_1. Next, one checks that there are no root of type 2 or 3. This implies r must be of type 1 or 4. The only mirrors of type 1 are the thirteen simple mirrors corresponding to $\mathcal{P}$. Now we make a list of those type 4 roots that satisfy $\mathrm{ht}(r) < 2.2$ (which is enough in

j	$\langle w_{\mathcal{P}}, r\rangle$	possible $(\langle x_1, r\rangle, \cdots, \langle x_{13}, r\rangle)$	$\tilde{\mathrm{ht}}(r)$	$\min(\mathrm{ht}(r))$
1	0	$(3u_1, 0^{12})$	$-u_1$	1
2		$\theta(u_1, u_2, u_3, 0^{10})$	$\frac{1}{\theta}\sum_{i=1}^{3} u_i$	no root
3	θ	$(3u_1, \theta u_2, 0^{11})$	$\frac{1}{\theta}(-4-\sqrt{3}+u_1-\theta u_2)$	no root
4		$\theta(u_1, u_2, u_3, u_4, 0^9)$	$\frac{1}{\theta}(-4-\sqrt{3}+\sum_{i=1}^{4} u_i)$	1
5	3	$3(u_1, u_2, 0^{11})$	$4+\sqrt{3}-u_1-u_2$	$2+\sqrt{3}$
6		$(3u_1, \theta u_2, \theta u_3, \theta u_4, 0^9)$	$4+\sqrt{3}-u_1+\frac{1}{\theta}\sum_{i=2}^{4} u_i$	3.24
7		$\theta(u_1, \cdots, u_6, 0^7)$	$4+\sqrt{3}+\frac{1}{\theta}\sum_{i=1}^{6} u_i$	2.73

TABLE 1. In each row, we consider roots r of a certain form, determined by the possible inner products of r with $w_{\mathcal{P}}$ and with $x_1, \cdots, x_{13}$ as shown in the second and third column respectively. ($u_1, u_2, \cdots$ stand for sixth roots of unity.) The roots considered in the j–th row are called the roots of type j. The fourth column records $\tilde{\mathrm{ht}}(r) = \langle \bar{\rho}, r\rangle / |\bar{\rho}|^2$ for a root r of type j. The fifth column records the minimum possible absolute value of the entry in the fourth column as u_i's vary over the sixth roots of unity, that is, the minimum possible height of a root of type j.

view of inequality (12)). The list consists of the thirteen simple mirrors corresponding to $\mathcal{L}$ and the 104 mirrors corresponding to the roots of the form $\phi_x^{\pm}(l)$, where $x \in \mathcal{P}$, $y \in \mathcal{L}$ and x is incident on l (Experimentally, these are the only roots having height $|1+\xi|$.) We want to show that these 130 mirrors do not meet Δ_1 except for the cases described in the statement of lemma 4.11. This is checked on the computer as follows:

Let $P = \{[\bar{\rho} + s_2 p_1 + s_3 q] \colon s_j \in \mathbb{R}\} \subseteq \mathbb{P}(V)$. Then P contains the triangle Δ_1. If the complex numbers $\langle r, \bar{\rho}\rangle$, $\langle r, p_1\rangle$ and $\langle r, q\rangle$ all have the same argument then we find that $\mathrm{Re}(\langle r, \bar{\rho}\rangle) = \mathrm{Re}(\langle r, p_1\rangle) = \mathrm{Re}(\langle r, q\rangle) = |\bar{\rho}|^2$. So a convex combination of $\bar{\rho}$, p_1 and q cannot be orthogonal to r. Otherwise $r^{\perp} \cap P$ is a point in $\mathbb{P}(V)$ which can be found by solving two linear equations. We have checked that this point does not belong to Δ_1 except in the cases mentioned in the statement of the lemma. □

proof of lemma 4.9. Assume that $r_1 \in \mathcal{P}$, $r_2 \in \mathcal{L}$ and that ϕ_{r_1} and ϕ_{r_2} braid. Let

$$r_3 = \phi_{r_1}(r_2) = \phi_{r_2}^{-1}(r_1) = r_2 + r_1 \quad \text{and} \quad r_4 = \phi_{r_2}(r_1) = \phi_{r_1}^{-1}(-\omega r_2) = r_1 - \omega r_2.$$

It suffices to show that the mirrors $\{r_1^{\perp}, \cdots, r_4^{\perp}\}$ do not meet C'. For this, it is enough to show that $\mathrm{Conv}(q', \sigma_1')$ do not intersect these four mirrors (since the same argument applies to $\mathrm{Conv}(q', \sigma_2')$ by symmetry). For $k = 1, \cdots, 6$, let $\Delta_k' = \mathrm{Conv}(q', z_{k-1}, z_k)$. For $k = 1, 2, 3$,

$j \setminus k$	0	1	2	3	4	5	6	r_j
1	1	0	$\bar\omega$	$\bar\omega c$	$\bar\omega - i$	$-ic$	$-i$	r_1
2	$\bar\xi$	$\bar\xi c$	$\bar\xi - \bar\omega$	$-\bar\omega c$	$-\bar\omega$	0	$-\omega$	r_2
3	$1+\bar\xi$	$\bar\xi c$	$\bar\xi$	0	$-i$	$-ic$	$-i-\omega$	$r_1 + r_2$
4	$1+\xi$	ξc	$\xi - \omega$	$-\omega c$	$-\omega - i$	$-ic$	$-i + \bar\omega$	$r_1 - \omega r_2$
z_k	$\bar\rho$	p_1	$\phi_x(\bar\rho)$	$\phi_x(p_2)$	$\phi_x\phi_l(\bar\rho)$	$\phi_x\phi_l(p_1)$	$\phi_l\phi_x\phi_l(\bar\rho)$	

TABLE 2. The left and top margin gives the row and column numbers (denoted by j and k). The j–th row of right margin records r_j. The k–th column in the bottom margin records z_k. The entry in the j–th row and k–th column is $c_j^k = \langle z_k, r_j\rangle/|\bar\rho|^2$. Finally, $c = 1 + 3^{-1/2}$.

let $\Delta''_k = \mathrm{Conv}(q', z^-_{2k-1}, z^+_{2k-1})$. So

$$\mathrm{Conv}(q', \sigma'_1) \subseteq \big(\cup_{k=1}^3 \Delta''_k\big) \cup \big(\cup_{k=1}^6 \Delta'_k \setminus \{z_1, z_3, z_5\}\big).$$

In table 2, we have recorded some inner products that we are going to need. Looking at the table we make the following observations:

(1) The numbers c_j^0 are always nonzero. In each row, atmost one entry is zero.
(2) For all j and k, we have $\mathrm{Re}(\xi c_j^k) \geq 0$.
(3) For each k, there is an open half plane $P_k \subseteq \mathbb{C}$ such that $\{c_j^0, \cdots, c_j^6\} \subseteq P_k \cup \{0\}$.

Let $s_1, s_2, s_3 \in \mathbb{R}$ such that $s_i \geq 0$ and $s_1 + s_2 + s_3 = 1$.

Let $k \in \{1, \cdots, 6\}$ and let $w_k = s_1 q' + s_2 z_{k-1} + s_3 z_k \in \Delta'_k$. If $\langle w_k, z_j\rangle = 0$, then

$$\langle w_k, r_j\rangle/|\bar\rho|^2 = s_1 \epsilon c_j^0 + s_2 c_j^{k-1} + s_3 c_j^k = 0, \tag{15}$$

where c_j^k are given in table 2. The three numbers $\{\epsilon c_j^0, c_j^{k-1}, c_j^k\}$ belong to $P_k \cup \{0\}$ for some open half plane P_k. So a convex combination of these three numbers is zero only when $c_j^{k-1} = 0$ and $(s_1, s_2, s_3) = (0, 1, 0)$ or $c_j^k = 0$ and $(s_1, s_2, s_3) = (0, 0, 1)$. It follows that equation (15) holds if and only if the pair (w_k, r_j) is equal to (z_1, r_1), or (z_3, r_3), or (z_5, r_2). So the only intersection of $\cup_{k=0}^6 \Delta'_k$ with $\cup_{j=1}^4 r_j^\perp$ is at the points $\{z_1, z_3, z_5\}$.

Now let $k \in \{1, 2, 3\}$ and let $w_k = s_1 q' + s_2 z^-_{2k-1} + s_3 z^+_{2k-1} \in \Delta''_k$. Then

$$\langle w_k, r_j\rangle/|\bar\rho|^2 = s_1 \epsilon c_j^0 + s_2(c_j^{2k-1} + \epsilon c_j^{2k-2}) + s_3(c_j^{2k-1} + \epsilon c_j^{2k}). \tag{16}$$

Note that the three numbers $\{\epsilon c_j^0, c_j^{2k-1} + \epsilon c_j^{2k-2}, c_j^{2k-1} + \epsilon c_j^{2k}\}$ always belong to some open half plane P_k. So a convex combination of these cannot be zero, that is, $\cup_{k=1}^3 \Delta''_k$ does not intersect $\cup_{j=1}^4 r_j^\perp$. When ϕ_1 and ϕ_2 commutes, the calculations are much easier and are omitted. □

5. The fixed points of diagram automorphisms

Recall, from section 3, that the diagram automorphisms $\mathrm{L}_3(3)$ point-wise fix a two dimensional primitive sub-lattice F spanned by $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$. Let $z_0 = w_{\mathcal{P}} + \theta w_{\mathcal{L}}$. Let $\Gamma_F = \{g \in \mathbb{P}\,\mathrm{Aut}(L) \colon g(F) = F\}$.

5.1. **Theorem.** *(a) There is an isometry $\beta : \mathbb{P}_+(F^{\mathbb{C}}) \to \mathcal{H}^2$ such that $\beta(\bar\rho) = i$, $\beta(w_{\mathcal{P}}) = p$, $\beta(w_{\mathcal{L}}) = -p^{-1}$ and $\beta(z_0) = 0$. (See figure 4).*

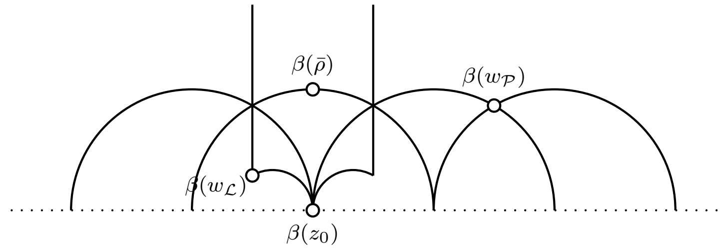

FIGURE 4.

(b) The isometry β induces a group homomorphism $c_\beta : \Gamma_F \to \mathrm{PSL}_2(\mathbb{R})$, given by $c_\beta(h) = \beta \circ h \circ \beta^{-1}$. The image of c_β contains the congruence subgroup $\Gamma(13) \subseteq \mathrm{PSL}_2(\mathbb{Z})$. In-fact

$$c_\beta(\Gamma_F) \cap \mathrm{PSL}_2(\mathbb{Z}) = \nu^{-1}\Gamma_0(13)\nu$$

where $\nu = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 5 \end{smallmatrix}\right)$. Further $c_\beta(\sigma) = S$.

5.2. *Remark.* The stabilizer in $\mathrm{Aut}(L)$ of the norm zero vector z_0 contains $\mathrm{L}_3(3)$. Theorem 4 of [3] implies that z_0 is a "cusp of Leech type", that is, $z_0^\perp/z_0$ is isomorphic to the complex Leech lattice.

5.3. *Remark.* Let Φ_L be the set of roots of L and $L_+^{\mathbb{C}}$ denote the set of vectors of $L^{\mathbb{C}}$ having positive norm. Consider $E_m : L_+^{\mathbb{C}} \to \mathbb{C}$, defined by,

$$E_m(z) = \sum_{r \in \Phi_L} \langle r, z \rangle^{-6m}.$$

Fix $z \in L_+^{\mathbb{C}}$. The number of roots r such that $|\langle r, z\rangle| \leq N$ grows as a polynomial in N. So the infinite sum above defines a non-constant meromorphic functions if m is large enough. The functions E_m are invariant under $\mathrm{Aut}(L)$ and have poles exactly along the mirrors of the reflection group $R(L)$. The lattice F is not contained in any mirror as there are no mirrors passing through $\bar\rho \in F^{\mathbb{C}}$. So the restriction of E_m to $F_+^{\mathbb{C}}$ is a non-constant meromorphic function invariant under Γ_F. Theorem 5.1 shows that Γ_F is commensurable with $\mathrm{PSL}_2(\mathbb{Z})$. So the restriction of E_m to $F_+^{\mathbb{C}}$ is a ordinary meromorphic modular form of level 13.

Starting with modular forms of singular weight with poles at cusps, one can construct meromorphic automorphic forms of type $U(1, n)$ by first taking Borcherds singular theta lift to get an automorphic form of type $O(2, 2n)$ and then restricting to the hermitian symmetric space of $U(1, n)$ (see theorem 7.1 of [1] and [6]). In our example, these Borcherds forms will be automorphic with respect to some finite index subgroup of $\mathrm{Aut}(L)$ and have their divisors along the mirrors, just like the functions $E_m(z)$. It will be interesting to understand how the restriction to $F_+^{\mathbb{C}}$ is related to the inverse of the Borcherds lift.

It follows from general considerations that the image of β is commensurable with $\mathrm{PSL}_2(\mathbb{Z})$. (This was explained to me by Stephen Kudla.) But we want to calculate the precise image. So We need to understand when an automorphism of F extend to an automorphism of L. For this, we need the following lemma. We urge the reader to recall, at this point, the construction of L from the diagram D given in section 3 and equations (2), (3) and (4).

5.4. **Lemma.** *Fix any $x_0 \in \mathcal{P}$ and $l_0 \in \mathcal{L}$. Let $\bar{L} = L/(F \oplus F^{\perp})$. Let*

$$13\bar{x} = \Sigma_{\mathcal{P}} = \sum_{x \in \mathcal{P}} x.$$

Let $\pi_F : L^{\mathbb{C}} \to F^{\mathbb{C}}$ and $\pi_{F^\perp} : L^{\mathbb{C}} \to (F^{\perp})^{\mathbb{C}}$ be the orthogonal projections.

(a) The lattice $F^{\perp}$ is spanned by the vectors $\{(x - x_0) \colon x \in \mathcal{P}\} \cup \{(l - l_0) : l \in \mathcal{L}\}$.

(b) One has $\Sigma_{\mathcal{P}} = 4w_{\mathcal{P}} - \bar{p}w_{\mathcal{L}} \in F$ and $(13x_0 - \Sigma_{\mathcal{P}}) \in F^{\perp}$. So $\pi_F(x_0) = \bar{x}$ and $\pi_{F^\perp}(x_0) = x_0 - \bar{x}$.

(c) One has an isomorphism $\mathcal{E}/13\mathcal{E} \simeq \bar{L}$ obtained by sending 1 to the image of x_0 (or l_0).

(d) Let $i_F : \bar{L} \to \mathrm{D}(F) = F^{\perp}/F$ be the injection given by $u \mapsto \pi_F(u) \bmod F$. Similarly define $i_{F^\perp} : \bar{L} \hookrightarrow \mathrm{D}(F^{\perp})$. It follows from part (b) and (c) that $i_F(\bar{L})$ is generated by $\bar{x}$ and $i_{F^\perp}(\bar{L})$ is generated by $(x_0 - \bar{x})$. The quotients $\mathrm{D}(F)/i_F(\bar{L})$ and $\mathrm{D}(F^{\perp})/i_{F^\perp}(\bar{L})$ are 3–groups.

(e) Given $g_1 \in \mathrm{Aut}(F)$ and $g_2 \in \mathrm{Aut}(F^{\perp})$, one can extend $(g_1 \times g_2)$ to an automorphism of L if and only if

$$i_F^{-1} \circ g_1 \circ i_F = i_{F^\perp}^{-1} \circ g_2 \circ i_{F^\perp}.$$

(In other words, g_1 and g_2 acts on the image of $\bar{F}$ in the same way).

(f) The automorphism σ acts on $i_{F^\perp}(\bar{L}) \subseteq \mathrm{D}(F^{\perp})$ as multiplication by $3\bar{p}$.

Proof. (a) Let $\tilde{F} = \mathrm{span}\{x - x_0, l - l_0 \colon x \in \mathcal{P}, l \in \mathcal{L}\}$. Equation (4) implies $\tilde{F} \subseteq F^{\perp}$. Since L is spanned by $\mathcal{P} \cup \mathcal{L}$, it is also spanned by $\tilde{F}$ together with x_0 and l_0. Given $w \in F^{\perp}$, we can write $w = \alpha_1 x_0 + \alpha_2 l_0 + \tilde{w}$, where $\tilde{w} \in \tilde{F}$. Taking inner product with $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$, one finds that $\alpha_1 = \alpha_2 = 0$, so $w \in \tilde{F}$. This proves part (a).

(b) For each of the four lines l passing through x_0, we have $w_{\mathcal{P}} = \bar{p}l + \sum_{x \in l} x$ (cf. equation (3)). Adding these together we get

$$4w_{\mathcal{P}} = \bar{p} \sum_{x_0 \in l} l + 3x_0 + \Sigma_{\mathcal{P}} = \bar{p}w_{\mathcal{L}} + \Sigma_{\mathcal{P}}.$$

So $\Sigma_{\mathcal{P}} \in F$. Clearly $13x_0 - \Sigma_{\mathcal{P}} = \sum_{x \in \mathcal{P}} (x_0 - x) \in F^{\perp}$.

(c) Note that $\bar{p}l_0 + 4x_0 = w_{\mathcal{P}} + \sum_{x \in l_0} (x_0 - x) \in F \oplus F^{\perp}$. Similarly $px_0 + 4l_0 \in F \oplus F^{\perp}$. So

$$l_0 - 3px_0 = (px_0 + 4l_0) - p(\bar{p}l_0 + 4x_0) \in F \oplus F^{\perp}. \tag{17}$$

So $F \oplus F^{\perp}$ together with x_0 generate L. Part (b) implies that $13x_0 \in F \oplus F^{\perp}$. Conversely suppose $\alpha x_0 \in F \oplus F^{\perp}$. Choose $u, v \in \mathcal{E}$ such that $(\alpha x_0 - uw_{\mathcal{P}} - vw_{\mathcal{L}}) \in F^{\perp}$. Taking inner product with $w_{\mathcal{L}}$ and $w_{\mathcal{P}}$ yield $\alpha\bar{p} = 4\bar{p}u + 3v$ and $0 = 3u + 4pv$. So

$$-13pv = p(4\bar{p}u + 3v) - 4(3u + 4pv) = 3\alpha,$$

that is, α is a multiple of 13.

(d) By calculating discriminants we find that $|\mathrm{D}(F)| = 13^2.3^2$ and $|\mathrm{D}(F^{\perp})| = 13^2.3^{12}$. Since $|\bar{L}| = 13^2$, it follows that $\mathrm{D}(F)/i_F(\bar{L})$ and $\mathrm{D}(F^{\perp})/i_{F^\perp}(\bar{L})$ are 3–groups.

(e) Note that $i_F(\bar{L})$ is generated by $\pi_F(x_0) = \bar{x}$ and $i_{F^\perp}(\bar{L})$ is generated by $\pi_{F^\perp}(x_0) = x_0 - \bar{x}$. From part (d) and Chinese remainder theorem we find that $\mathrm{D}(F)$ is the direct product of $i_F(\bar{L})$ and a 3–group. So g_1 acts on $i_F(\bar{L})$ as multiplication by some $\lambda \in (\mathcal{E}/13\mathcal{E})^*$ (since $\bar{L}$ is a cyclic $\mathcal{E}$–module and g_1 is $\mathcal{E}$–linear). Similarly, as $\mathrm{D}(F^{\perp})$ is the product of $i_{F^\perp}(\bar{L})$ and a 3–group, g_2 acts on $i_{F^\perp}(\bar{L})$ as multiplication by some λ'. The condition $i_F^{-1} \circ g_1 \circ i_F = i_{F^\perp}^{-1} \circ g_2 \circ i_{F^\perp}$ is equivalent to $\lambda = \lambda'$. If this condition is satisfied, we have

$$(g_1 \times g_2)(x_0) = g_1(\bar{x}) + g_2(x_0 - \bar{x}) \equiv \lambda\bar{x} + \lambda(x_0 - \bar{x}) \bmod F \oplus F^{\perp} \equiv \lambda x_0 \bmod F \oplus F^{\perp},$$

which implies that $(g_1 \times g_2)(x_0) \in L$. The converse is also clear.

(f) Assume that $\sigma(l_0) = x_0$. Let $\bar{l} = \sum_{l \in \mathcal{L}} l/13$. Then $\pi_F(l_0) = \bar{l}$. We have seen in equation (17) that $v = l_0 - 3px_0 \in F \oplus F^{\perp}$. So

$$v - \pi_F(v) = (l_0 - 3px_0) - (\bar{l} - 3p\bar{x}) \in F^{\perp},$$

that is, $(l_0 - \bar{l}) \equiv 3p(x_0 - \bar{x}) \bmod F^{\perp}$. It follows that

$$\sigma(x_0 - \bar{x}) = -\omega(l_0 - \bar{l}) \equiv -\omega(3p)(x_0 - \bar{x}) \bmod F^{\perp}. \qquad \square$$

5.5. **Definition** (of the map β)**.** The group $Q \subseteq \Gamma$ of diagram automorphisms fixes two lines in $L^{\mathbb{C}}$. (see 5.5 of [4]). These are the lines containing the vectors $\bar{\rho}_+$ and $\bar{\rho}_-$, where

$$\bar{\rho}_{\pm} = \frac{w_{\mathcal{P}} \pm \xi w_{\mathcal{L}}}{2(4 \pm \sqrt{3})}.$$

So $F^{\mathbb{C}} = \bar{\rho}_+\mathbb{C} \oplus \bar{\rho}_-\mathbb{C}$. Let $n_{\pm} = \sqrt{\pm|\bar{\rho}_{\pm}|^2}$. Recall (cf. [4], section 5.4 and 5.5) that

$$n_{\pm}^2 = \frac{\sqrt{3}}{2(4 \pm \sqrt{3})}.$$

Note that $|p \pm i|^2 = 4 \pm \sqrt{3}$, so $\frac{n_+^2}{n_-^2} = \frac{4-\sqrt{3}}{4+\sqrt{3}} = \frac{|p-i|^2}{|p+i|^2}$. Define $\alpha \in [0, 2\pi)$ by $\frac{p-i}{p+i} = \left(\frac{n_+}{n_-}\right)e^{i\alpha}$. (The angle α is chosen so that the isometry β mentioned in theorem 5.1 takes $w_{\mathcal{P}}$ and $w_{\mathcal{L}}$ to $\mathrm{SL}_2(\mathbb{Z})$ conjugates of ω.)

Consider the basis of $F^{\mathbb{C}}$ given by $y_1 = e^{-i\alpha}\bar{\rho}_-/n_-$ and $y_2 = \bar{\rho}_+/n_+$. Observe that $-|y_1|^2 = |y_2|^2 = 1$ and $\langle y_1, y_2 \rangle = 0$. So $\beta' : \mathbb{P}_+(F^{\mathbb{C}}) \to B^1(\mathbb{C})$ given by $\beta'[uy_1 + vy_2] = \frac{v}{u}$ is an isometry. Changing basis, we get,

$$\beta'[u\bar{\rho}_+ + v\bar{\rho}_-] = e^{i\alpha}\left(\tfrac{n_-}{n_+}\right)\left(\tfrac{v}{u}\right).$$

Let $\beta = C \circ \beta' : P_+(F^{\mathbb{C}}) \to \mathcal{H}^2$ where $C(w) = i\left(\frac{1+w}{1-w}\right)$ is the Cayley isomorphism. The map β is an isometry since both β' and C are. Some calculation using the identities given above, yields $\beta'[aw_{\mathcal{P}} + bw_{\mathcal{L}}] = \left(\frac{p-i}{p+i}\right)\left(\frac{a-\bar{\xi}b}{a+\bar{\xi}b}\right)$. Composing with the Cayley isomorphism and simplifying, one gets,

$$\beta[aw_{\mathcal{P}} + bw_{\mathcal{L}}] = \frac{pa + \omega b}{a + \bar{p}b} \quad \text{and} \quad \beta^{-1}(\tau) = [(1 + p\tau)w_{\mathcal{P}} + \omega^2(\tau - p)w_{\mathcal{L}}]. \tag{18}$$

Matrix notation: For the rest of this section, we identify $z = uw_{\mathcal{P}} + vw_{\mathcal{L}} \in F^{\mathbb{C}}$ with the column vector $\begin{pmatrix} u \\ v \end{pmatrix}$. Accordingly, the isometry β is represented by the matrix $\begin{pmatrix} p & \omega \\ 1 & \bar{p} \end{pmatrix}$, which we again denote by β by a slight abuse of notation.

5.6. **Lemma.** *(a) Let* $J_F = \mathrm{gram}(F)_w = \begin{pmatrix} 3 & 4p \\ 4\bar{p} & 3 \end{pmatrix}$ *(see equation (4)). Then*

$$J_F = \bar{\theta}\beta^* S \beta. \tag{19}$$

Let $\beta z = \left(\begin{smallmatrix} \tau_1 \\ \tau_2 \end{smallmatrix}\right)$, $\beta z' = \left(\begin{smallmatrix} \tau_1' \\ \tau_2' \end{smallmatrix}\right)$. *Then* $\langle z, z' \rangle = \bar{\theta}(\tau_1'\bar{\tau}_2 - \tau_2'\bar{\tau}_1)$. *In particular* $|z|^2 = 2\sqrt{3}\,\mathrm{Im}(\tau_1\bar{\tau}_2)$.
(b) Let $\nu = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 5 \end{smallmatrix}\right)$. *Given* $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, *let* $g_1 = \beta^{-1}g\beta$. *Then* $(z \mapsto g_1 z)$ *is an isometry of the Hermitian vector space* $F^{\mathbb{C}}$. *Further,* $g_1 \in \mathrm{Aut}(F)$ *if and only if*

$$g \in \nu^{-1}\Gamma_0(13)\nu = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \colon \begin{pmatrix} a - d \\ b + c \end{pmatrix} \in \mathbb{Z}\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}.$$

Proof. The equation (19) is verified by multiplying matrices. Given $z, z' \in F^{\mathbb{C}}$, one has

$$\langle z, z'\rangle = z^* J_F z' = \bar\theta z^* \beta^* S \beta z' = \bar\theta (\bar\tau_1 \;\; \bar\tau_2) S \left(\begin{smallmatrix} \tau_1' \\ \tau_2' \end{smallmatrix}\right) = \bar\theta(\tau_1'\bar\tau_2 - \tau_2'\bar\tau_1).$$

(b) To show that g_1 is an isometry we calculate as follows:

$$\langle g_1 z_1, g_1 z_2\rangle = z_1^* \beta^* g^* (\beta^{-1})^* J_F \beta^{-1} g \beta z_2 = \bar\theta z_1^* \beta^* g^* S g \beta z_2 = \bar\theta z_1^* \beta^* S \beta z_2 = z_1^* J_F z_2 = \langle z_1, z_2\rangle.$$

The second equality uses (19) and the third one follows from $g^{tr} S g = S$ for all $g \in \mathrm{Sp}_2(\mathbb{Z})$.

Let $p_1 = \det(\beta) = 3 - \omega$. Then $p_1 \bar p_1 = 13$. Let $a - d = s$ and $b + c = t$. We find that

$$g_1 = \frac{1}{p_1}\begin{pmatrix} p_1 a + \omega s + \bar p t & -\omega p_1 b - \bar\omega p s - \bar\omega t \\ -\bar\omega p_1 c - ps - t & p_1 d - \omega s - \bar p t \end{pmatrix}.$$

Note that $\omega s + \bar p t \equiv ps + t \bmod p_1$. So $g_1 \in M_2(\mathcal{E})$ if and only if $p_1 \mid (ps + t)$. This is equivalent to the condition

$$\begin{pmatrix} s \\ t \end{pmatrix} \in \mathbb{Z}\begin{pmatrix} 3 \\ -2 \end{pmatrix} + \mathbb{Z}\begin{pmatrix} 2 \\ 3 \end{pmatrix} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{Z}^2 \colon 5x + y \equiv 0 \bmod 13 \right\}.$$

So $g_1 \in M_2(\mathcal{E})$ if and only if g has the form $\left(\begin{smallmatrix} d+s & b \\ t-b & d \end{smallmatrix}\right)$ with $5s + t \equiv 0 \bmod 13$. On the other hand, we find that

$$\nu \begin{pmatrix} d+s & b \\ t-b & d \end{pmatrix} \nu^{-1} \equiv \begin{pmatrix} * & * \\ -5s - t & * \end{pmatrix} \bmod 13. \qquad \square$$

proof of theorem 5.1. We have already constructed the required isometry β. The claims $\beta(\bar\rho) = i$, $\beta(w_{\mathcal{P}}) = -\beta(w_{\mathcal{L}})^{-1} = p$ and $\beta(w_{\mathcal{P}} + \theta w_{\mathcal{L}}) = 0$ are easily checked using (18). The automorphism σ interchanges $w_{\mathcal{P}} = \beta^{-1}(p)$ and $w_{\mathcal{L}} = \beta^{-1}(-p^{-1})$ and fixes $\bar\rho = \beta^{-1}(i)$. So $c_\beta(\sigma) = S$.

Suppose $g \in \mathrm{SL}_2(\mathbb{Z})$ such that $\nu g \nu^{-1} \in \Gamma_0(13)$. Lemma 5.6(b) shows that $g_1 = \beta^{-1} g \beta$ is an isometry of the lattice F. Since $i_F(\bar L)$ is generated by $\bar x$ (see 5.4 (d)), there exists $\lambda \in (\mathcal{E}/13\mathcal{E})^*$ such that $g_1(\bar x) \equiv \lambda \bar x \bmod F$. From 5.4(e) we find that g_1 can be extended to an automorphism of L if there exists an automorphism of $F^\perp$ that acts on $i_{F^\perp}(\bar L)$ as multiplication by λ. In particular, 5.4(f) implies that g_1 can be extended to L if λ belongs to the multiplicative subgroup of $(\mathcal{E}/13\mathcal{E})^*$ generated by $3\bar p$. We check this by a direct calculation, sketched below.

Recall from 5.4(b) that $\pi_F(x) = \Sigma_{\mathcal{P}}/13$, where $\Sigma_{\mathcal{P}} = 4w_{\mathcal{P}} - \bar p w_{\mathcal{L}} \in F$. So in our matrix notation, $\Sigma_{\mathcal{P}} = \left(\begin{smallmatrix} 4 \\ -\bar p \end{smallmatrix}\right)$. Observe that $\beta\left(\begin{smallmatrix} 4 \\ -\bar p \end{smallmatrix}\right) = \det(\beta)\left(\begin{smallmatrix} p \\ -\bar\omega \end{smallmatrix}\right)$ and

$$g \equiv \begin{pmatrix} 2k + d & b \\ 3k - b & d \end{pmatrix} \bmod 13,$$

(since $\left(\begin{smallmatrix} 3 \\ -2 \end{smallmatrix}\right) \equiv -5\left(\begin{smallmatrix} 2 \\ 3 \end{smallmatrix}\right) \bmod 13$). So

$$\begin{aligned} g_1 \begin{pmatrix} 4 \\ -\bar p \end{pmatrix} &\equiv \beta^{adj}\begin{pmatrix} 2k & 0 \\ 3k & 0 \end{pmatrix}\begin{pmatrix} p \\ -\bar\omega \end{pmatrix} + \beta^{adj}\begin{pmatrix} d & b \\ -b & d \end{pmatrix}\begin{pmatrix} p \\ -\bar\omega \end{pmatrix} \\ &\equiv 3k(4 + 3\omega)\begin{pmatrix} 4 \\ -\bar p \end{pmatrix} + d\begin{pmatrix} 4 \\ -\bar p \end{pmatrix} + \bar\omega b\begin{pmatrix} -p \\ 4 \end{pmatrix} \\ &\equiv (3k(4 + 3\omega) + d + 3p\bar\omega b)\begin{pmatrix} 4 \\ -\bar p \end{pmatrix} \bmod 13, \end{aligned}$$

where the last congruence follows by observing that $\binom{-p}{4} \equiv \binom{12p}{-9} \equiv 3p\binom{4}{-\bar{p}} \bmod 13$. In other words $g_1(\bar{x}) \equiv \lambda\bar{x} \bmod F$ where

$$\lambda = [3k(4+3\omega) + d + 3p\bar{\omega}b] \bmod 13\mathcal{E}.$$

To finish the argument, we have to show that λ belongs to the multiplicative group generated by $3p$ in $\mathcal{E}/13\mathcal{E}$. Recall the prime factorization $13 = p_1\bar{p}_1$, (where $p_1 = 3-\omega$). By Chinese remainder theorem, we have an isomorphism

$$\varphi : \mathcal{E}/13\mathcal{E} \to \mathcal{E}/p_1\mathcal{E} \oplus \mathcal{E}/\bar{p}_1\mathcal{E} \simeq \mathbb{F}_{13} \times \mathbb{F}_{13}, \text{ given by, } x \mapsto (x \bmod p_1, x \bmod \bar{p}_1).$$

Now, using $\varphi(\omega) = (3,9)$ and $\varphi(\bar{\omega}) = (9,3)$, it follows that

$$\varphi(3p) = (2,7) = (2, 2^{-1}) \in \mathbb{F}_{13}^* \times \mathbb{F}_{13}^* \text{ and } \varphi(\lambda) = (d+5b, 2k+d+8b).$$

Since 2 is a generator of $\mathbb{F}_{13}^*$, it follows that λ belong to the multiplicative group generated by $3p$ if and only if $\varphi(\lambda)$ has the form (u,v) with $u.v = 1$. To finish the proof, we note that

$$(d+5b)(2k+d+8b) \equiv k(2d-3b) + d^2 + b^2 \equiv \det(g) \equiv 1 \bmod 13. \qquad \square$$

5.7. **Lemma.** *(a) Let $z \in F$ be a norm 3 vector. Then there exists $g_z \in \mathrm{SL}_2(\mathbb{Z})$ such that $\beta(z) = g_z(\omega)$.*
(b) Let $r \in F$ be a vector of norm -3. Let $z \in F \cap r^\perp$ be a primitive non-zero vector. Then there exists $g_z \in \mathrm{SL}_2(\mathbb{Z})$ such that $\beta(z) = g_z(\omega)$.

Proof. (a) Let $z \in F$. Let us write $\beta.z = \binom{s}{t} = \binom{s_1\omega+s_2}{t_1\omega+t_2}$ with $s_j, t_j \in \mathbb{Z}$. Part (a) of lemma 5.6 implies that $|z|^2 = 3$ if and only if $\sqrt{3} = 2\,\mathrm{Im}(\bar{t}s)$, which simplifies to $s_1t_2 - s_2t_1 = 1$. So we can take $g_z = \begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix}$.
(b) Let $\beta.r = \binom{s}{t}$. Now $|r|^2 = -3$ implies $2\,\mathrm{Im}(\bar{t}s) = -\sqrt{3}$. So $2\,\mathrm{Im}(\bar{s}\bar{\bar{t}}) = \sqrt{3}$. As in part (a), this implies $\bar{s}/\bar{t} \in \mathrm{PSL}_2(\mathbb{Z})\omega$. Now let $\beta.z = \binom{u}{v}$. One has $0 = \langle z, r\rangle = \bar{\theta}(-\bar{s}v + \bar{t}u)$. It follows that $\beta(z) = u/v = \bar{s}/\bar{t} \in \mathrm{PSL}_2(\mathbb{Z})\omega$. $\square$

5.8. *Remark.* The stabilizer of $\omega \in \mathcal{H}^2$ in $\mathrm{PSL}_2(\mathbb{Z})$ is the $\mathbb{Z}/3\mathbb{Z}$ generated by ST. If $z \in F$ has norm 3, then the $120°$ rotation around $z \in \mathbb{P}_+(F)$ belongs to Γ_F. If $r \in F$ is a vector of norm -3 and z is a primitive vector in $F \cap r^\perp$, then the restriction of ϕ_r^ω to $\mathbb{P}_+(F)$ is an element of Γ_F, which is again a $120°$ rotation around z. Lemma 5.7 implies that the image of these rotations under c_β are $\mathrm{PSL}_2(\mathbb{Z})$ conjugates of ST or $(ST)^{-1}$.

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